Probabilistic Deontic Logics

Master Artificial Intelligence Thesis

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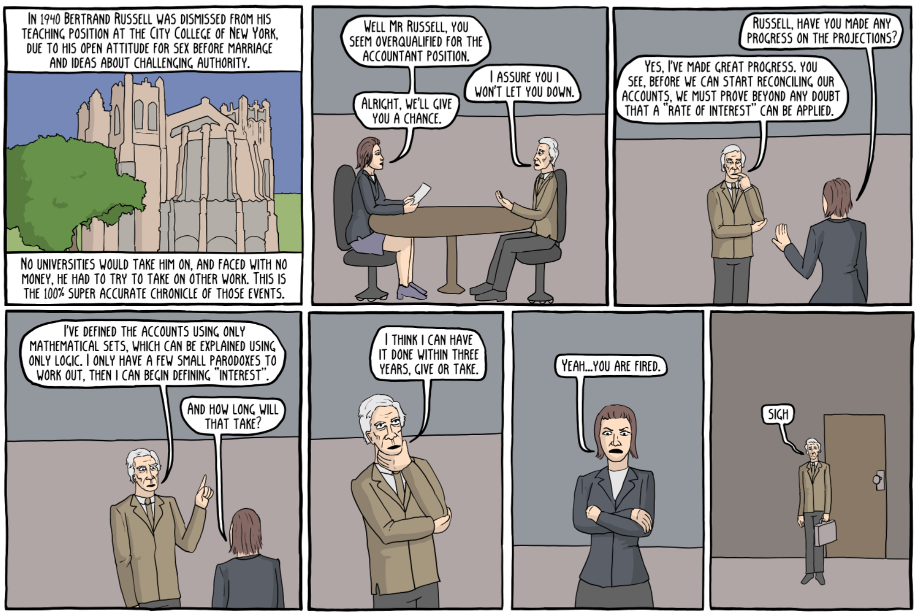

Abstract

Normative reasoning plays a role in thinking about how things should or should not be. Such perceptions occur everywhere from the personal to the organizational scale. In some situations there is uncertainty about norms for example currently there can be uncertainty about whether having to shake hands. In this thesis, we develop several logical formalism's for representing and reasoning about uncertainty in regard to norms. Where uncertainty is represented by probability and norms are represented by deontics. Both concepts have their own logic and the idea is to combine the two into one logic. First a logic will be developed that is able to express probability about normative statements, like "the probability that one is obliged to shake hands is 0.5". Afterwards the formalism is extended to represent an agent's uncertainty about normative statements but also about other agent's their uncertainties. Then we will go even further by considering the same uncertainty paradigms for conditional obligations. For every logic multiple definitions will be given, namely: Language, Models, Satisfaction, the Axiomatization & proof of Soundness and Completeness. **Keywords: Deontic Logic, Probabilistic Logic, Uncertainty, Obligation, Formalization**

Contents

1 Introduction

Artificial Intelligence is the field that attempts to simulate intelligence. The field has many sub fields of which some more popular than others. At the moment of writing Machine Learning has ushered in a new AI hype. Before Machine Learning the hypes were brought about by Logic. Still the AI sub field of Knowledge Representation and Reasoning develops. But what happens when Artificial Intelligence is actually created? Will all of humanity be exterminated, as some movies seem to imply? Underneath the ideas on which these movies are based lies a struggle between ethical versus lawful. When an AI is created we can ask ourselves the question should it be ethical or should it be lawful? An ethical AI might classify humans to be unethical and thereby decide to exterminate them. While a lawful AI might do the same based on loopholes in the law. Of course it can also be that the AI, however it is implemented, thinks well of people and wants to work with them in either of the situations. Furthermore when we try to solve it by saying that the AI should be both then what happens when ethical rules and lawful rules clash? Notions that capture this type of thoughts are uncertainty and obligation. For both of these notions logics exist that formalize them namely respectively: Probabilistic Logic and Deontic Logic.

More specifically Deontic Logic is the logic that formalizes deontics; "that which is binding or proper". The Deontic Logics used in this project are Monadic and Dyadic Deontic Logic. These logics are modal logics and make use of a language similar to propositional logic. The Probabilistic Logic used in this project utilizes a propositional language and a Kripke structure which is extended with weight formulas depicting probability. Also two types of Probabilistic Logic will be introduced, their most interesting difference is their probability structure. Deontic and Probabilistic Logic are used to formalize respectively the concepts: obligation and uncertainty. Namely Deontic Logic formalizes obligation and Probabilistic Logic formalizes certainty. The goal of this project is to combine these logics into one logic Probabilistic Deontic Logic: $\mathfrak{PDL}$. This logic will then be able to capture the described notion: uncertainty about obligations.

1.1 Related Work

Previous work already shows two promising methods by combining Probability theory with a formalization of Deontics. In the paper "A probabilistic deontic argumentation framework" probability and deontics are used in an argumentation framework [9]. In this paper two argumentation frameworks are merged one probabilistic and the other deontic. The probabilistic argumentation framework encapsulates the deontic argumentation framework. The deontic argumentation framework is is ASPIC+-like and adds deontic principles to this. ASPIC+ is an argumentation framework that is meant to generate abstract argumentation frameworks in the sense of [14]. The probabilistic argumentation framework associates probability to different argument labelings. These methods are combined into a framework in which different labelings of a deontics containing argument are assigned a probability. Another paper titled "Learning Behavioral Norms in Uncertain and Changing Contexts" [10] specifies a probabilistic deontic logic for use in relation to robots learning norms. The logic uses the deontic concepts: Obligation, Forbidden, and Permitted. Furthermore each deontic formula is able to take a subscript for the circumstances. For example to denote that it is forbidden to talk in a library, the subscript library can be used: $F_{library}talk$. The probability of a formula is denoted as a superscript to the deontic symbol, with a lower and an upper bound. The lower bound is related to the belief in the formula and the upper bound relates to the plausibility of the formula: $F^{[0.7,1]}_{library}talk$. In this way probabilistic deontic formula's are represented in [10]. Even though these approaches combine deontics and probability one is in argumentation theory and not plain logic and the other does not give a thorough formalization.

1.2 Overview of project

The plan for this project is to construct a logic that is able to describe uncertainty towards obligations. This will be done with regards to the following topics:

- Syntax & Semantics
    - Formulae
    - Model
    - Satisfaction
- Axiomatization
- Soundness & Completeness

Also Deontic Logics and Probabilistic Logics will be introduced with regards to the same topics. Except a proof of soundness & completeness will not be given, since this can be found in the literature. After the introduction of the logics four constructions of $\mathfrak{PDL}$will be given. Then after this in the Conclusion & Discussion section design decisions and possible further research will be pointed out.

Please note that the logics are introduced with notation close to their usual notations. In the section that defines $\mathfrak{PDL}$ we will introduce modified notation which will be useful to keep objects with similar properties apart. Furthermore we denote with $\mathbb{P}$ the set of atom propositions and with $\mathbb{Q}$ the set of rational numbers with is denoted; which is used to keep the set of weight formulas countable.

2 Background Work

In this section two logics will be introduced. The logics are Deontic Logic and Probabilistic Logic. Of both logics two variants will be introduced. These logics are relevant to this project because they formalize the notions that we want to combine: obligation and uncertainty.

2.1 Deontic Logic

Deontic logic is the logic that formalizes the old greek concept Deontic. This concept can be interpreted in multiple ways but the conventional meaning is Obligation or Ought [13]. There are multiple logics that can be seen as Deontic Logic: Monadic Deontic Logic, Dyadic Deontic Logic, Input Output logic, Standard Deontic Logic, etc. Here two modal logics will be given in detail namely: Monadic Deontic Logic and Dyadic Deontic Logic.

2.1.1 Monadic Deontic Logic

Monadic Deontic Logic is a modal logic and introduces the deontic operator O, it uses O much alike the necessary operator $\Box$ but with a different axiomatization [15]. The operator can be bound to a formula ϕ: $O\phi$ meaning ϕ is obligatory, the language is given by the following definitions. The O operator is different from $\Box$ in axiomatization in that it includes the D axiom: $O\phi \to P\phi$.

Syntax & **Semantics**

Definition 1 (Formulae). *The language $\mathcal{L}$ of monadic deontic logic is generated by the following BNF (Backus Normal Form):*

$$[\mathcal{L}_{MDL}] \quad \phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid O\phi \quad p \in \mathbb{P}$$

The construct $O\phi$ is read as "It is obligatory that ϕ." Other connectives are derivable:

disjunction	$\phi \vee \phi'$	*is*	$\neg(\neg\phi \wedge \neg\phi')$
implication	$\phi \to \phi'$	*is*	$(\neg\phi) \vee \phi'$
equivalence	$\phi \leftrightarrow \phi'$	*is*	$(\phi \to \phi') \wedge (\phi' \to \phi)$
verum	$\top$	*is*	$\phi \vee \neg\phi$
falsum	$\bot$	*is*	$\neg\top$
permission	$P\phi$	*is*	$\neg O \neg\phi$
prohibition	$F\phi$	*is*	$O\neg\phi$

The above Backus-Naur Form (BNF) is a convenient abbreviation for the following alternative definition. The language of monadic deontic logic is the smallest set $\mathcal{L}$ such that:

- $\mathbb{P} \subseteq \mathcal{L}$
- *If $\phi \in \mathcal{L}$, then $\neg\phi \in \mathcal{L}$ and $O\phi \in \mathcal{L}$*

- *If $\phi \in \mathcal{L}$ and $\psi \in \mathcal{L}$, then $\phi \wedge \psi \in \mathcal{L}$*

A model in MDL is given by the following definition.

Definition 2 (Relational model). *A relational model $\mathcal{M}_{MDL}$ is a tuple $\mathcal{M}_{MDL} = (W, R, V)$ where:*

- *W is a (non-empty) set of states (also called "possible worlds"); W is called the universe of the model.*
- *$R \subseteq W \times W$ is a binary relation over W. It is understood as a relation of deontic alternativeness: sRt (or, alternatively, $(s,t) \in R$) says that t is an ideal alternative to s, or that t is a "good" successor of s. The first one is "good" in the sense that it complies with all the obligations true in the second one. Furthermore, R is subject to the following constraint:*

  $$(\forall s \in W)(\exists t \in W)(sRt) \qquad \textit{(seriality)}$$

  This means that the model does not have a dead end, a state with no good successor.
- *$V : \mathbb{P} \mapsto 2^W$ is a valuation function assigning to each atom p a set $V(p) \subseteq W$ (intuitively the set of states at which p is true.)*

The following definition gives defines satisfaction of formulas inside the just defined models.

Definition 3 (Satisfaction). *Given a relational model $\mathcal{M}_{MDL} = (W, R, V)$ and a state $s \in W$, we define the satisfaction relation $\mathcal{M}_{MDL}, s \models_{MDL} \phi$ (read as "state s satisfies ϕ in model $\mathcal{M}_{MDL}$", or as "s makes ϕ true $\mathcal{M}_{MDL}$") by induction on the structure of ϕ using the following clauses:*

- *$\mathcal{M}_{MDL}, s \models_{MDL} p$ iff $s \in V(p)$*
- *$\mathcal{M}_{MDL}, s \models_{MDL} \neg\phi$ iff $\mathcal{M}_{MDL}, s \not\models_{MDL} \phi$*
- *$\mathcal{M}_{MDL}, s \models_{MDL} \phi \wedge \psi$ iff $\mathcal{M}_{MDL}, s \models_{MDL} \phi$ and $\mathcal{M}_{MDL}, s \models_{MDL} \psi$*
- *$\mathcal{M}_{MDL}, s \models_{MDL} O\phi$ iff for all $t \in W$, if sRt then $\mathcal{M}_{MDL}, t \models_{MDL} \phi$*

Axiomatization The axiom system commonly used in Monadic Deontic Logic is the following system [8] referred to as system **D**.

Propositional Reasoning:

PL. $\vdash \phi$, where ϕ is a propositional tautology

MP. If $\vdash \phi$ and $\vdash \phi \rightarrow \psi$ then $\vdash \psi$

Reasoning with O:

O-K. $\vdash O(\phi \rightarrow \psi) \rightarrow (O\phi \rightarrow O\psi)$

O-D. $\vdash O\phi \rightarrow P\phi$

O-Nec. If $\vdash \phi$ then $\vdash O\phi$

The axiom system uses two axioms for Propositional reasoning: PL. and MP. Then to axiomatize the operator O three axioms are used, two of these are standard modal axioms that can be find for all modal operators which are: O-K. and O-Nec. In this case the axioms are written using the O operator when describing the alethic $\Box$ operator one would instead of O use $\Box$. The extra axiom needed to axiomatize O is $O - D$ and describes that if something is obligatory it is also permitted.

2.1.2 Dyadic Deontic Logic

Monadic Deontic Logic as explained uses the alternativeness relation R to model which states are good alternatives of other states. But this system is only able to express situations in which all obligations are met versus situations where not all obligations are met. Alternatively it would be good to be able to rank alternative states where the best states are the ones where all obligations are met and the worst states are ones where none of the obligations are met. Monadic Deontic Logic can be modified to be able to deal with this problem. This is done in Dyadic Deontic Logic this is a non-normal version of deontic logic. In this logic the O is treated differently, namely as a conditional operator. Unconditional obligations can still be expressed through $O(\phi|\top)$. Therefore DDL is viewed as a generalization of MDL. Where MDL uses a valid successor relation (for a state s if sRt then t is a valid successor) to make the relation ought (O) explicit in the models. DDL requires, due to the change to a conditional obligation operator, a different relation between states. This is due to the possibility of an agent not following a primary obligation (ie. $O(\phi|\top)$) while succeeding to follow a contrary to duty obligation (ie. $O(\psi|\neg\phi)$). When the primary obligation is not fulfilled the state is not an optimal successor, though when the contrary to duty obligation is fulfilled the state is preferred over one in which the contrary to duty obligation is also not fulfilled. Ranking the possible states in this way determines which states are optimal in DDL. The logic DDL is given by the following definitions.

Syntax & Semantics

Definition 4 (Formulae)**.** *The Language of DDL is generated by the following BNF:*

$$[\mathcal{L}_{DDL}] \quad \phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid \Box\phi \mid \Diamond\phi \mid O(\phi|\phi) \quad p \in \mathbb{P}$$

$\Box\phi$ is read as "ϕ is settled as true", and $O(\psi|\phi)$ as "ψ is obligatory, given ϕ". ϕ is called the antecedent, and ψ the consequent. $P(\psi|\phi)$ ("ψ is permitted, given ϕ") is short for $\neg O(\neg\psi|\phi)$, $O\phi$ ("ϕ is unconditionally obligatory") and $P\phi$ ("ϕ is unconditionally permitted") are short for $O(\phi|\top)$ and $P(\phi|\top)$, respectively. $\Diamond\phi$ is short for $\neg\Box\neg\phi$. Other Boolean connectives are as defined in definition 1.

The DDL model is given by the following definition.

Definition 5 (Preference model)**.** *A preference model $\mathcal{M}_{DDL}$ is a tuple $\mathcal{M}_{DDL} = (W, \succeq, V)$ where:*

- *W is a (non-empty) set of states (also called "possible worlds"). W is called the universe of the model.*

- $\succeq$ *is a binary relation over* W *ordering the states according to their relative goodness.* $s \succeq t$ *is read as "state* s *is at least as good as state* t*".*
- $V : \mathbb{P} \mapsto 2^W$ *is a valuation function assigning to each atom* p *a set* $V(p) \subseteq W$ *(intuitively the set of states at which* p *is true.)*

The language and models of DDL are connected through satisfaction of formulas in states of the model. The satisfaction relation is defined as follows.

Definition 6 (Satisfaction)**.** *Given a preference model* $\mathcal{M}_{DDL} = (W, \succeq, V)$ *and a state* $s \in W$*, we define the satisfaction relation* $\mathcal{M}_{DDL}, s \models_{DDL} \phi$ *as:*

- $\mathcal{M}_{DDL}, s \models_{DDL} p$ *iff* $s \in V(p)$
- $\mathcal{M}_{DDL}, s \models_{DDL} \neg\phi$ *iff* $\mathcal{M}_{DDL}, s \not\models_{DDL} \phi$
- $\mathcal{M}_{DDL}, s \models_{DDL} \phi \wedge \psi$ *iff* $\mathcal{M}_{DDL}, s \models_{DDL} \phi$ *and* $\mathcal{M}_{DDL}, s \models_{DDL} \psi$
- $\mathcal{M}_{DDL}, s \models_{DDL} \Box\phi$ *iff, for all* $t \in W$*,* $\mathcal{M}_{DDL}, t \models_{DDL} \phi$
- $\mathcal{M}_{DDL}, s \models_{DDL} O(\psi|\phi)$ *iff* $best(\|\phi\|) \subseteq \|\psi\|$

where $\|\phi\|$ *is the set of states that satisfy* ϕ *ie.* $\|\phi\| = \{s \in W \mid \mathcal{M}_{DDL}, s \models_{DDL} \phi\}$*. And* $best(A)$ *the subset of states in* $A \subseteq W$ *that are optimal according to* $\succeq$*:* $best(A) = opt_{\succeq}(A) = \{s \in A \mid \forall t \in A \quad s \succeq t\}$*.*

Lastly the preference relation has a few properties that should properly be introduced. The properties are given by the following definition.

Definition 7 (Properties of $\succeq$)**.**

reflexivity:	$\forall s$	$s \succeq s$
transitivity:	$\forall s, t, u$	$s \succeq t \wedge t \succeq u \rightarrow s \succeq u$
totalness:	$\forall s, t$	$s \succeq t \vee t \succeq s$
limitedness:		$\|\phi\| \neq \emptyset \rightarrow opt_{\succeq}(\|\phi\|) \neq \emptyset$

These properties are given in [8] to describe the relation $\succeq$. Intuitively these properties make sure that all states of a model can be compared with each other. Also it assures, with the limitedness property, that infinite sequences of better than (no best) are ruled out.

Axiomatization Lastly an axiomatization of the proof system of DDL. DDL has multiple proof systems; three systems introduced in [8] are **E**, **F** and **G**. The following system of axioms is system **E** this is the basic proof system, **F** adds an extra axiom to it and **G** adds an extra axiom to **F**.

Propositional reasoning:

PL. ϕ, where ϕ is a tautology from PL

MP. If $\vdash \phi$ and $\vdash \phi \rightarrow \psi$ then $\vdash \psi$

Reasoning with $\Box$:

$\Box$-K. $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$

$\Box$-T. $\Box\phi \rightarrow \phi$

□-5. $\neg\Box\phi \to \Box\neg\Box\phi$

□-Nec. If $\vdash \phi$ then $\vdash \Box\phi$

Reasoning with $O(-|-)$:

COK. $O(\psi \to \chi|\phi) \to (O(\psi|\phi) \to O(\chi|\phi))$

id. $O(\phi|\phi)$

Sh. $O(\chi|(\phi \wedge \psi)) \to O((\psi \to \chi)|\phi)$

Interplay of □ and $O(-|-)$:

Abs. $O(\psi|\phi) \to \Box O(\psi|\phi)$

Nec. $\Box\psi \to O(\psi|\phi)$

Ext. $\Box(\phi \leftrightarrow \psi) \to (O(\chi|\phi) \leftrightarrow O(\chi|\psi))$

System **F** is formed by adding the rule (O-D*) to system **E**. This axiom is useful since it rules out the possibility of conflicts between conditional obligations.

(O-D*) $\Diamond\phi \to (O(\psi|\phi) \to P(\psi|\phi))$

And system **G** is formed by adding the rule (Sp) to system **F**. This axiom is equivalent to the principle of rational monotony: $(P(\psi|\phi) \wedge O(\chi|\phi) \to O(\chi|\phi \wedge \psi))$.

(Sp) $P(\psi|\phi) \wedge O((\psi \to \chi)|\phi) \to O(\chi|(\phi \wedge \psi))$

Since system **E** is conventionally used it will be used in the construction of $\mathfrak{PDL}$. System **F** or **G** could also be used given the straightforward modification of the axiom system and semantics.

2.2 Probabilistic Logic

Probabilistic Logic is a logic that allows probability into the language. Probability can be used to describe multiple things, one of the nice properties of probability is that the total amount of probability is 1, furthermore probabilities are non-negative. In this project probability is used to model uncertainty. In this section two probabilistic logics are introduced these are [2] and [1]. Both logics use the same type of probability measure, but they have a different probability structure. Now the probability measure will be introduced since both logics use it. But before the probability measure can be given two concepts need to be defined.

Definition 8 (σ-algebra)**.** *The σ-algebra of a set S is a collection $\mathscr{X}$ of subsets of S that includes S itself, is closed under complement, and is closed under countable union. The power set is an example of a σ-algebra.*

Definition 9 (Measurable model). *A model M is* measurable *if*

$$\|\phi\|^M \in \mathscr{X}$$

for every $\phi \in \mathcal{L}$.

Where $\mathcal{L}$ is the language of model M and $\mathscr{X}$ is defined on the set of states S in M. Now using the σ-algebra and notion of measurable model we can define a Standard Probability Measure.

Definition 10 (Standard Probability Space). *A Standard Probability Space is a tuple $(S, \mathscr{X}, \mu)$. Where S is a set of states or possible worlds. $\mathscr{X}$ is a σ-algebra of subsets of S whose elements are called measurable sets. And μ is a standard probability measure defined on the measurable sets. Thus $\mu : \mathscr{X} \to [0, 1]$ and satisfying the following properties:*

P1. $\mu(X) \geq 0$ *for all* $X \in \mathscr{X}$

P2. $\mu(S) = 1$

P3. $\mu(\bigcup_{i=1}^{\infty} X_i) = \sum_{i=1}^{\infty} \mu(X_i)$, *if the X_i's are pairwise disjoint.*

Now the logics can be further given using this definition of a Standard Probability Measure.

2.2.1 Type 1 Probabilistic Logic

The logic introduced in the paper "A Logic for Reasoning about Probabilities" [2] introduces a logic which can be used to reason about probabilities, this is achieved using weight functions. A primitive weight term is a term that wraps the function $w()$ around a propositional formula ϕ: $w(\phi)$. A weight term scales such terms and can consist of a sum involving multiple terms: $a_1 w(\phi_1) + ... + a_k w(\phi_k)$. A basic weight formula is an inequality involving a weight term: $a_1 w(\phi_1) + ... + a_k w(\phi_k) \geq \alpha$.

Syntax & Semantics

Definition 11 (Formulae). *Let Φ denote the set of propositional formulas which is the set $\mathbb{P}$ closed under conjunction and negation. The Language of the Type 1 Probabilistic Logic is generated by the following BNF, with $a_1, \ldots, a_k, \alpha \in \mathbb{Q}$:*

$$[\mathcal{L}_{T1-prob}] \quad f ::= a_1 w(\phi_1) + \cdots + a_n w(\phi_n) \geq \alpha \mid \neg f \mid (f \wedge f) \quad \phi \in \Phi$$

The construct $w()$ in the construct $w(\phi) \geq \alpha$ stands for "weight" and depicts the probability (certainty/uncertainty) with respect to a formula ϕ. The language also gives rise to other connectives namely:

disjunction	$f \vee f'$	*is*	$\neg(\neg f \wedge \neg f')$
implication	$f \rightarrow f'$	*is*	$(\neg f) \vee f'$
equivalence	$f \leftrightarrow f'$	*is*	$(f \rightarrow f') \wedge (f' \rightarrow f)$
verum	$\top$	*is*	$f \vee \neg f$
falsum	$\perp$	*is*	$\neg\top$
lesser or equal than	$\sum_{i=1}^{n} a_i w(\phi_i) \leq \alpha$	*is*	$\sum_{i=1}^{n} -a_i w(\phi_i) \geq -\alpha$
equality	$\sum_{i=1}^{n} a_i w(\phi_i) = \alpha$	*is*	$\sum_{i=1}^{n} a_i w(\phi_i) \geq \alpha \bigwedge \sum_{i=1}^{n} a_i w(\phi_i) \leq \alpha$
strictly greater than	$\sum_{i=1}^{n} a_i w(\phi_i) > \alpha$	*is*	$\neg(\sum_{i=1}^{n} a_i w(\phi_i) \leq \alpha)$
strictly lesser than	$\sum_{i=1}^{n} a_i w(\phi_i) < \alpha$	*is*	$\neg(\sum_{i=1}^{n} a_i w(\phi_i) \geq \alpha)$

Definition 12 (Type 1 - Model). *A probability structure $\mathscr{M}_{T1}$ [2] is a tuple $\mathscr{M}_{T1} = (S, \mathscr{X}, \mu, \pi)$ using a standard probability measure, where:*

- *S is a set of states or possible worlds.*
- *$\mathscr{X}$ is a σ-algebra of subsets of S whose elements are called measurable sets.*
- *μ is a probability measure defined on the measurable sets. Thus $\mu : \mathscr{X} \rightarrow [0, 1]$.*
- *π associates with each state in S a truth assignment on the primitive propositions in $\mathbb{P}$. Thus $\pi(s)(p) \in \{$**true, false**$\}$, for each $s \in S$ and $p \in \mathbb{P}$. Furthermore $\pi(s)$ is extended in the usual way to evaluate truth of all propositional formulas.*

Definition 13 (Satisfaction). *A model satisfies a weight formula in the following circumstances, with $a_1, \ldots, a_k, \alpha \in \mathbb{Q}$:*

- $(\mathscr{M}_{T1}, s) \models \phi \wedge \psi$ *iff* $(\mathscr{M}_{T1}, s) \models \phi$ *and* $(\mathscr{M}_{T1}, s) \models \psi$
- $(\mathscr{M}_{T1}, s) \models \neg\phi$ *iff* $(\mathscr{M}_{T1}, s) \not\models \phi$
- $\mathscr{M}_{T1} \models a_1 w(\phi_1) + ... + a_k w(\phi_k) \geq \alpha$ *iff* $a_1 \mu(\phi_1^{\mathscr{M}_{T1}}) + ... + a_k \mu(\phi_k^{\mathscr{M}_{T1}}) \geq \alpha$

Where $\phi^{\mathscr{M}_{T1}}$ is given by

$$\phi^{\mathscr{M}_{T1}} = \{s \in S \mid \pi(s)(\phi) = \textbf{\textit{true}}\}.$$

Axiomatization The paper [2] also provides a complete axiomatization named $AX_{MEAS-T1}$. This system of axioms gives a sound and complete axiomatization of valid formulas in the measurable case. The formulas f and g both are formulas of $\mathcal{L}_{T1-prob}$, and ϕ and ψ are formulas of Φ

Propositional reasoning:

PL. All instances of propositional tautologies

MP. From f and $f \rightarrow g$ infer g (modus ponens).

Reasoning about linear inequalities:

I1. $x \geq x$ (identity)

I2. $(a_1x_1 + ... + a_kx_k \geq c) \leftrightarrow (a_1x_1 + ... + a_kx_k + 0x_{k+1} \geq c)$ (adding and deleting 0 terms)

I3. $(a_1x_1 + ... + a_kx_k \geq c) \rightarrow (a_{j_1}x_{j_1} + ... + a_{j_k}x_{j_k} \geq c)$, if $j_1, ..., j_k$ is a permutation of $1, ..., k$ (permutation)

I4. $(a_1x_1 + ... + a_kx_k \geq c) \wedge (a'_1x_1 + ... + a'_kx_k \geq c') \rightarrow ((a_1 + a'_1)x_1 + ... + (a_k + a'_k)x_k \geq (c + c'))$ (addition of coefficients)

I5. $(a_1x_1 + ... + a_kx_k \geq c) \leftrightarrow (da_1x_1 + ... + da_kx_k \geq dc)$ if $d > 0$ (multiplication of nonzero coefficients)

I6. $(t \geq c) \vee (t \leq c)$ if t is a term (dichotomy)

I7. $(t \geq c) \rightarrow (t > d)$ if t is a term and $c > d$ (monotonicity)

Reasoning about probabilities:

W1. $w(\phi) \geq 0$ (nonnegativity).

W2. $w(true) = 1$ (the probability of the event **true** is 1).

W3. $w(\phi \wedge \psi) + w(\phi \wedge \neg\psi) = w(\phi)$ (additivity).

W4. $w(\phi) = w(\psi)$ if $\phi \leftrightarrow \psi$ is a propositional tautology (distributivity).

W5. $w(false) = 0$ (the probability of the event **false** is 0).

Of these axioms W1, W2 and W3 correspond to P1, P2 and P3 of Definition 12.

2.2.2 Type 2 Probabilistic Logic

In "Reasoning about knowledge and probability" [1] the knowledge operator K is used to create sentences like $K_i\phi$: i knows that ϕ is the case, (with i denoting agent i). But also in combination with weight formulas $K_i(w_i(\phi) \geq \alpha)$ which is also written compactly as $K_i^\alpha(\phi)$: i knows that the probability of ϕ is atleast α. The language introduced in [1] allows nesting this means that ϕ can be a formula containing weight functions and/or knowledge operators. This means the following formulas are possible $K_i(w_j(\phi) \geq \alpha)$ (i knows that for j the probability of ϕ is atleast α), $w_i(K_j\phi) \geq \alpha$ (for i the probability that j knows ϕ is atleast α), $K_jK_i\phi$ (j knows that i knows that ϕ is the case), $w_i(w_j(\phi) \geq \alpha) \geq \beta$) (for i the probability that for j the probability of ϕ is atleast α is atleast β) but also even deeper iterations.

Syntax & **Semantics**

Definition 14 (Formulae)**.** *Let Φ denote the set of propositional formulas which is the set $\mathbb{P}$ closed under conjunction and negation. The Language of the Type 2 Probabilistic Logic is generated by the following BNF, with $a_1, \ldots, a_k, \alpha \in \mathbb{Q}$:*

$$[\mathcal{L}_{T2-prob}] \quad f ::= \phi \mid a_1 w_i(f_1) + \cdots + a_n w_i(f_n) \geq \alpha \mid \neg f \mid f \wedge f \mid K_i f \qquad \phi \in \Phi$$

The construct $w_{i,s}()$ in the construct $w_{i,s}(f) \geq \alpha$ stands for "weight" and depicts the probability (certainty/uncertainty) attributed by agent i in state s with respect to a formula f. The language also gives rise to other connectives equivalent to definition 11.

Definition 15 (Type 2 - Model)**.** *A Kripke structure $\mathscr{M}_{T2}$ for knowledge and probability (for n agents) is a tuple $\mathscr{M}_{T2} = (S, \pi, \mathscr{K}_1, \ldots, \mathscr{K}_n, \mathscr{P})$ using standard probability measure, where*

- *S is a set of states or possible worlds.*
- *π associates with each state in S a truth assignment on the primitive propositions in Φ. Thus $\pi(s)(p) \in \{$**true, false**$\}$, for each $s \in S$ and $p \in \Phi$.*
- *$\mathscr{K}_i$ is an equivalence relation on the states of S, for $i = 1, \ldots, n$. The relation is intended to capture the possibility relation according to agent $i : (s,t) \in \mathscr{K}_i$ if in world s agent i considers t a possible world. We define $\mathscr{K}_i(s) = \{s' \mid (s, s') \in \mathscr{K}_i\}$.*
- *$\mathscr{P}$ is a probability assignment, which assigns to each agent $i \in \{1, ..., n\}$ and state $s \in S$ a probability space $\mathscr{P}(i,s) = (S_{i,s}, \mathscr{X}_{i,s}, \mu_{i,s})$ where*
  - *$S_{i,s} \subseteq S$ a non-empty subset of S.*
  - *$\mathscr{X}_{i,s}$ is a σ-algebra of subsets of $S_{i,s}$ whose elements are called measurable sets.*
  - *$\mu_{i,s}$ is a probability measure defined on the measurable sets. Thus $\mu_{i,s} : \mathscr{X}_{i,s} \to [0, 1]$.*

Definition 15 shows the assignment of a probability space to each combination of state and agent. The assignment consists of a set of considered states $S_{i,s}$, a σ-algebra set on that set of states $\mathscr{X}_{i,s}$ and a probability measure $\mu_{i,s}$.

Definition 16 (Satisfaction)**.** *Satisfaction of a weight formula in a Type 2 Model is defined as follows using induction:*

- *$(\mathscr{M}_{T2}, s) \models p$ (for $p \in \Phi$) iff $\pi(s)(p) =$ **true***
- *$(\mathscr{M}_{T2}, s) \models f \wedge g$ iff $(\mathscr{M}_{T2}, s) \models f$ and $(\mathscr{M}_{T2}, s) \models g$*
- *$(\mathscr{M}_{T2}, s) \models \neg f$ iff $(\mathscr{M}_{T2}, s) \not\models f$*
- *$(\mathscr{M}_{T2}, s) \models K_i f$ iff $(\mathscr{M}_{T2}, t) \models f$ for all $t \in \mathscr{K}_i(s)$*
- *$(\mathscr{M}_{T2}, s) \models a_1 w_i(f_1) + ... + a_k w_i(f_k) \geq \alpha$ iff $a_1 \mu_{i,s}(S_{i,s}(f_1)) + ... + a_k \mu_{i,s}(S_{i,s}(f_k)) \geq \alpha$*

With $S_{i,s}(f) = \{s' \in S_{i,s} \mid (\mathscr{M}_{T2}, s') \models f\}$.

Axiomatization For the sake of completeness provided are axioms used in this logic for usage of the K operator. The axiomatization $AX_{MEAS-T2}$ is an expansion and modification of the axiomatization of the logic introduced in the previous subsection; ie. Definition 2.2.1 type 1 probabilistic logic. Epistemic axioms are added in this axiomatization and agent subscripts. Later when constructing a $\mathfrak{PDL}$variant with this type of logic the epistemic part will be removed. The formulas f and g both are formulas of $\mathcal{L}_{T2-prob}$.

Propositional reasoning:

PL. All instances of propositional tautologies

MP. From f and $f \rightarrow g$ infer g (modus ponens).

Reasoning about linear inequalities:

I1.-I7. see axiomatization in Section 2.2.1

Reasoning about probabilities:

W1. $w_i(f) \geq 0$ (nonnegativity).

W2. $w_i(true) = 1$ (the probability of the event **true** is 1).

W3. $w_i(f \wedge g) + w_i(f \wedge \neg g) = w_i(f)$ (additivity).

W4. $w_i(f) = w(g)$ if $f \leftrightarrow g$ is a propositional tautology (distributivity).

W5. $w_i(false) = 0$ (the probability of the event **false** is 0).

Reasoning about knowledge:

K2. $(K_i f \wedge K_i(f \rightarrow g)) \rightarrow K_i g$.

K3. $K_i f \rightarrow f$.

K4. $K_i f \rightarrow K_i K_i f$.

K5. $\neg K_i f \rightarrow K_i \neg K_i f$.

N. From f infer $K_i f$ (knowledge generalization).

3 Probabilistic Deontic Logics

The goal of this section is to formalize a Probabilistic Deontic Logic ($\mathfrak{PDL}$) that is able to express uncertainty about obligations. The logics that will be used for this combination are the logics introduced in the previous section, Section 2. Those logics are Monadic and Dyadic Deontic Logic and two types of Probabilistic Logic. Four $\mathfrak{PDL}$constructions will be made, each of them involving one Deontic Logic and one type of Probabilistic Logic. Examples of how to combine the logics can be drawn from [2, 1].

The notion that will be expressed by $\mathfrak{PDL}$is uncertainty about obligations. The first construction will involve MDL and the type 1 Probabilistic Logic. This means that the construction should be able to formalize sentences of the following form: $w(O\phi) \geq \alpha$. Furthermore due to nesting we should consider whether $O(w(\phi) \geq \alpha)$ is allowed into the logic. Nesting of operators will be restricted to the extent that sentences of type $O(w(\phi) \geq \alpha)$ ie. Obligation that ϕ is uncertain, are prohibited. The reason for this is that the primary sentences that should be expressible are of the type $w(O\phi) \geq \alpha$. When $O(w(\phi) \geq \alpha)$ is allowed into the logic both of the formulas need to be interpretable under the same semantics. This is difficult to guarantee, this topic will be further discussed in Section 4. In later constructions the logic will be modified to allow (1) uncertainty about someone else's uncertainty and (2) uncertainty about conditional obligations.

In the coming subsections four model constructions will be presented. The four constructions are only different with regards to the logics they combine. The four combinations are constructions possible from the previously explained four types of logic. Namely MDL or DDL combined with Probabilistic logic with a Type 1 or Type 2 probability structures, this results in four combinations. To combine these logics a function will be used, τ, that relates the deontic worlds to probabilistic states. In each of the combinations τ is a pivotal part of the model construction. This function will allow the content of deontic worlds –a deontic state will be called a world for disambiguation purposes– into the states of the probabilistic model. Due to the way τ works, the type of possible sentences is constrained. Namely the obligation operator can not take a probabilistic formula as an argument, as explained earlier is preferred. Lastly throughout the section rational numbers are used ie. numbers from $\mathbb{Q}$ this keeps the number of possible formulas involving $w()$ countable.

Each of the following subsections will be dedicated to one of the possible combinations for constructing a probabilistic deontic logic. In the subsections definitions and or re-definitions will be given. Also it is possible that a definition will be reiterated for the sake of comfort, or that large parts that do not change will be left out. Furthermore the topics that will be discussed in each subsection are:

- Syntax & Semantics
  - Formulae
  - Model
  - Satisfaction
- Axiomatization

- Soundness & Completeness

An example will be used to illustrate each logic and to have a baseline of comparison for the different constructions. The example is about the relation between ethical and lawful rules.

3.1 Type 1 Monadic $\mathfrak{PDL}$

The first $\mathfrak{PDL}$ construction we discuss is a construction of MDL with a probability structure of type 1 probabilistic logic. This is relatively the simplest construction since it uses the simpler probability structure and the normal modal deontic logic. This construction serves to some extent as a starting point from which we can develop the other constructions. This is due to the central idea (τ) used in the construction being introduced here next to other constitutional definitions. We will shorten the name Type 1 Monadic Probabilistic Deontic Logic to M1-$\mathfrak{PDL}$, later constructions will be abbreviated in an equivalent way.

Syntax and Semantics In this section the syntax and semantics of this initial $\mathfrak{PDL}$construction is presented. This logic contains two types of formulas: standard deontic formulas of MDL, and probabilistic formulas based on the ones found in [2]. Let $\mathbb{Q}$ denote the set of rational numbers, which is used to keep the set of probabilistic formulas countable.

Definition 17 (Formulae)**.** *Let $\mathbb{P}$ be a set of atomic propositions. The language $\mathcal{L}_{M1-\mathfrak{PDL}}$ of type 1 monadic probabilistic deontic logic is generated by the following two sentences of BNF (Backus Naur Form):*

$$
\begin{array}{lll}
[\mathcal{L}_{MDL}] & \phi ::= p \mid \neg\phi \mid (\phi \wedge \phi) \mid O\phi & p \in \mathbb{P} \\
[\mathcal{L}_{M1-\mathfrak{PDL}}] & f ::= a_1 w(\phi_1) + \cdots + a_n w(\phi_n) \geq \alpha \mid \neg f \mid (f \wedge f) &
\end{array}
$$

The construct $O\phi$ is read as "It is obligatory that ϕ", and $w()$ in the construct $w(\phi) \geq \alpha$ stands for "weight" and depicts the probability (certainty/uncertainty) with respect to a deontic formula ϕ. Both languages also give rise to other *connectives namely:*

disjunction	$\theta \vee \theta'$	*is*	$\neg(\neg\theta \wedge \neg\theta')$
implication	$\theta \rightarrow \theta'$	*is*	$(\neg\theta) \vee \theta'$
equivalence	$\theta \leftrightarrow \theta'$	*is*	$(\theta \rightarrow \theta') \wedge (\theta' \rightarrow \theta)$
verum	$\top$	*is*	$\theta \vee \neg\theta$
falsum	$\bot$	*is*	$\neg\top$
permission	$P\phi$	*is*	$\neg O \neg\phi$
prohibition	$F\phi$	*is*	$O\neg\phi$
lesser or equal than	$\sum_{i=1}^{n} a_i w(\phi_i) \leq \alpha$	*is*	$\sum_{i=1}^{n} -a_i w(\phi_i) \geq -\alpha$
equality	$\sum_{i=1}^{n} a_i w(\phi_i) = \alpha$	*is*	$\sum_{i=1}^{n} a_i w(\phi_i) \geq \alpha \bigwedge \sum_{i=1}^{n} a_i w(\phi_i) \leq \alpha$
strictly greater than	$\sum_{i=1}^{n} a_i w(\phi_i) > \alpha$	*is*	$\neg \sum_{i=1}^{n} a_i w(\phi_i) \leq \alpha$
strictly lesser than	$\sum_{i=1}^{n} a_i w(\phi_i) < \alpha$	*is*	$\neg \sum_{i=1}^{n} a_i w(\phi_i) \geq \alpha$

Note that with θ are denoted either formulas $\phi \in \mathcal{L}_{MDL}$ or formulas $f \in \mathcal{L}_{M1-\mathfrak{PDL}}$. Mixing of the formulas from $\mathcal{L}_{MDL}$ and $\mathcal{L}_{M1-\mathfrak{PDL}}$ is not included in the language. For example, $O(p \vee q) \wedge w(Oq) \geq 0.9$ (p or q is obligatory and the probability that q is obligatory is 0.9) is not a formula of the language. The reason for using two languages is that the formulas ϕ come from MDL. These formulas "live" in MDL models and are incorporated into a probabilistic model therefore the two languages do not interact in the way described. The language of M1-$\mathfrak{PDL}$does however use the language of MDL as arguments for its weight terms. Now the semantics of M1-$\mathfrak{PDL}$is introduced.

Example 1. *Following the example introduced above about rules of ethics and law, the fact that a rules is considered established might be expressed by the probabilistic statement "the probability that one is obliged to protect life is at least 0.9". This sentence could be formalized using the introduced language as*

$$w(Oq) \geq 0.9.$$

Definition 18 (Model). *A M1-$\mathfrak{PDL}$model is a tuple $M = \langle S, \mathscr{X}, \mu, \tau \rangle$, which combines monadic deontic logic models with a type 1 probability structure:*

- *S is a non-empty set of* states
- *$\mathscr{X}$ is a σ-algebra of subsets of S*
- *$\mu : \mathscr{X} \rightarrow [0, 1]$ is a standard probability measure as defined in Definition 10.*
- *τ associates with each state in S a tuple containing a monadic deontic model and one of its worlds: $\tau(s) \mapsto (D_s, w_s)$ where:*

- *$D_s = (W_s, R_s, V_s)$ is a model of monadic deontic logic.*
- *$w_s \in W_s$ is a world w_s in W_s of model D_s.*

Example 1. ***(continued)*** *Assume a finite set of atomic propositions $\{p, q\}$. Let us consider the model $M = \langle S, \mathscr{X}, \mu, \tau \rangle$, where*

- *$S = \{s, s', s'', s'''\}$*
- *$\mathscr{X}$ is the set of all subsets of S*
- *μ is characterized by: $\mu(\{s\}) = 0.5$, $\mu(\{s'\}) = \mu(\{s''\}) = 0.2$, $\mu(\{s'''\}) = 0.1$ (other values follow from the properties of probability measures)*
- *τ is a mapping which assigns to the state s, $D_s = (W_s, R_s, V_s)$ and w_s such that*
  - *$W_s = \{w_1, w_2, w_3, w_4\}$*
  - *$R_s = \{(w_1, w_2), (w_1, w_3), (w_2, w_2), (w_2, w_3), (w_3, w_2), (w_3, w_3), (w_4, w_2), (w_4, w_3), (w_4, w_4)\}$*
  - *$V_s(p) = \{w_1, w_3\}$, $V_s(q) = \{w_2, w_3\}$*
  - *$w_s = w_1$*

  Note that the domain of τ is always the whole set S, but in this example we only explicitly specify $\tau(s)$ for illustration purposes.

This model is depicted in Figure 1. The circle on the right contains the four states of the model, which are measured by μ. Each of the states is equipped with a standard pointed model of MDL. In this picture, only one of them is shown, the one that corresponds to s. It ($\tau(s)$) is represented within the circle on the left. Note that the arrows depict the "good" alternative relation R. If we assume that q stands for "protect life", like in the previous example, in all good successors of w_1 the proposition q holds. Note that, according to Definition 3, this means that agents in w_1 are obliged to protect life.

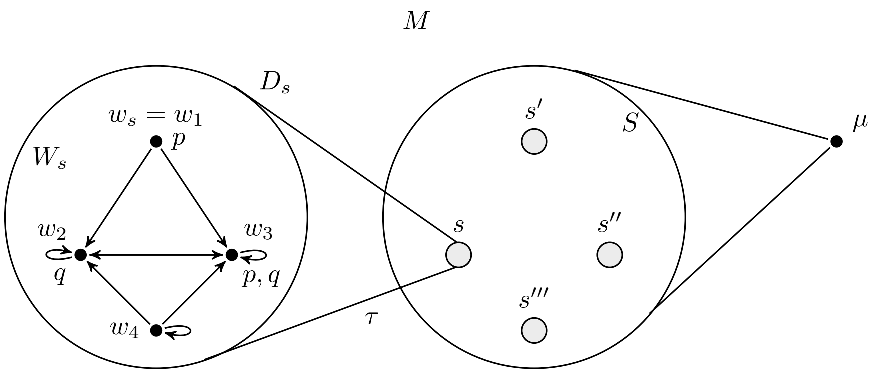

Figure 1: Model $M = \langle S, \mathscr{X}, \mu, \tau \rangle$.

Next the satisfiability of a formula in a model can be defined. First the truth of a deontic formula in a state of a $\mathfrak{PDL}$model is given. This definition is in accordance with the standard satisfiability relation of MDL $\models_{MDL}$.

Definition 19 (Satisfaction)**.** *Let $M = \langle S, \mathscr{X}, \mu, \tau \rangle$ be a measurable probabilistic deontic model. We define the satisfiability relation $\models$ recursively as follows:*

- $M \models \phi$ *iff for all* $s \in S$, $(D_s, w_s) \models_{MDL} \phi$
- $M \models a_1 w(\phi_1) + \cdots + a_k w(\phi_k) \geq \alpha$ *iff* $a_1 \mu(\|\phi_1\|) + \cdots + a_k \mu(\|\phi_k\|) \geq \alpha$.
- $M \models \neg f$ *iff* $M \not\models f$
- $M \models f \wedge g$ *iff* $M \models f$ *and* $M \models g$

For a model $M = \langle S, \mathscr{X}, \mu, \tau \rangle$ and a formula $\phi \in \mathcal{L}_{MDL}$, let $\|\phi\|^M$ denote the set of states that satisfy ϕ, i.e.,

$$\|\phi\|^M = \{s \in S \mid M, s \models \phi\}.$$

We omit the superscript M from $\|\phi\|^M$ when it is clear from context. The following definition introduces an important class of probabilistic deontic models, so called measurable *models.*

The focus is on measurable structures and completeness is proven for this class of structures.

Definition 20 (Measurable model)**.** *A probabilistic deontic model is* measurable *if*

$$\|\phi\|^M \in \mathscr{X}$$

for every $\phi \in \mathcal{L}_{MDL}$.

Example 1. ***(continued)*** *Continuing the previous example, according to Definition 19 it holds that $M, s \models Oq$. At this point it is also possible to speak of the probability of the obligation to protect life. Assume that τ is defined in the way such that $M, s' \models Oq$ and $M, s'' \models Oq$, but $M, s''' \not\models Oq$. Then $\mu(\|Oq\|) = \mu(\{s, s', s''\}) = 0.5 + 0.2 + 0.2 = 0.9$. According to Definition 19, $M \models w(Oq) \geq 0.9$.*

Note that, according to Definition 19, a deontic formula is true in a model iff it holds in every state of the model. This is a consequence of our design choice that those formulas represent certain deontic knowledge, while probabilistic formulas express uncertainty about norms. Now some definitions of some standard semantical notions will be given.

Definition 21 (Semantical consequence)**.** *Given a set Γ of formulas, a formula θ is a semantical consequence of Γ (notation: $\Gamma \models \theta$) whenever all the states $s \in S$ of the model M have, if $M, s \models \theta'$ for all $\theta' \in \Gamma$, then $M, s \models \theta$.*

Definition 22 (Validity)**.** *A formula θ is valid (notations: $\models \theta$) whenever for $M = \langle S, \mathscr{X}, \mu, \tau \rangle$ and every $s \in S$: $M, s \models \theta$ holds.*

Axiomatization The following axiomatization combines the axioms of proof system D of monadic deontic logic [8] with the axioms of the type 1 probabilistic logic. The axioms for reasoning about linear inequalities are taken from [2]. The only axioms that occur in both the axiom systems of MDL and Probabilistic Logic are modus ponens and Tautologies. These two axioms are generalized to work for both types of formulas. The axioms of MDL have been added to the axiom system and depict axioms for deontic formulas ϕ. The axioms of type 1 probabilistic logic also have been added to the axiom system and are depicted using weight formulas f. The axioms show in this way the Deontic and Probabilistic deductions that can be made within the M1-$\mathfrak{PDL}$model.

Tautologies and Modus Ponens

Taut. All instances of propositional tautologies.

MP. From θ and $\theta \rightarrow \theta'$ infer θ'.

Reasoning with O:

O-K. $O(\phi \rightarrow \phi') \rightarrow (O\phi \rightarrow O\phi')$

O-D. $O\phi \rightarrow P\phi$

O-Nec. *From ϕ infer $O\phi$*

Reasoning about linear inequalities:

I1. $x \geq x$ (identity)

I2. $(a_1x_1 + ... + a_kx_k \geq c) \leftrightarrow (a_1x_1 + ... + a_kx_k + 0x_{k+1} \geq c)$ (adding and deleting 0 terms)

I3. $(a_1x_1 + ... + a_kx_k \geq c) \rightarrow (a_{j_1}x_{j_1} + ... + a_{j_k}x_{j_k} \geq c)$, if $j_1, ..., j_k$ is a permutation of $1, ..., k$ (permutation)

I4. $(a_1x_1 + ... + a_kx_k \geq c) \wedge (a'_1x_1 + ... + a'_kx_k \geq c') \rightarrow ((a_1 + a'_1)x_1 + ... + (a_k + a'_k)x_k \geq (c + c'))$ (addition of coefficients)

I5. $(a_1x_1 + ... + a_kx_k \geq c) \leftrightarrow (da_1x_1 + ... + da_kx_k \geq dc)$ if $d > 0$ (multiplication of nonzero coefficients)

I6. $(t \geq c) \vee (t \leq c)$ if t is a term (dichotomy)

I7. $(t \geq c) \rightarrow (t > d)$ if t is a term and $c > d$ (monotonicity)

Reasoning about probabilities:

W1. $w(\phi) \geq 0$ (nonnegativity).

W2. $w(\phi \vee \phi') = w(\phi) + w(\phi')$, if $\neg(\phi \wedge \phi')$ is an instance of a classical propositional tautology (finite additivity).

W3. $w(\top) = 1$

P-Dis. From $\phi \leftrightarrow \phi'$ infer $w(\phi) = w(\phi')$ (probabilistic distributivity)

The axiom Taut allows all $\mathcal{L}_{MDL}$-instances and $\mathcal{L}_{M1-\mathfrak{PDL}}$-instances of propositional tautologies. For example, $w(Oq) \geq 0.9 \vee \neg w(Oq) \geq 0.9$ is an instance of Taut, but $w(Oq) \geq 0.9 \vee \neg w(Oq) \geq 1$ is not. Note that Modus Ponens (MP) can be applied to both types of formulas, but only if θ and θ' are both from $\mathcal{L}_{MDL}$ or both from $\mathcal{L}_{M1-\mathfrak{PDL}}$. O-Nec is a deontic variant of necessitation rule. P-Dis is an inference rule which states that two equivalent deontic formulas must have the same probability values.

Definition 23 (Syntactical consequence)**.** *A derivation of θ is a finite sequence $\theta_1, \ldots, \theta_m$ of formulas such that $\theta_m = \theta$, and every θ_i is either an instance of an axiom, or it is obtained by the application of an inference rule to formulas in the sequence that appear before θ_i. If there is a derivation of θ, we say that θ is a* theorem *and write $\vdash \theta$. We also say that θ is derivable from a set of formulas Γ, and write $\Gamma \vdash \theta$, if there is a finite sequence $\theta_1, \ldots, \theta_m$ of formulas such that $\theta_m = \theta$, and every θ_i is either a theorem, a member of Γ, or the result of an application of MP. or P-Dis. to formulas in the sequence that appear before θ_i.*

Note that this definition restricts the application of O-Nec.. This is a standard restriction for modal necessitation, which enables one to prove Deduction theorem using induction on the length of the inference. Also, note that only deontic formulas can participate in a proof of another deontic formula, thus derivations of deontic formulas in our logic coincide with their derivations in MDL.

Definition 24 (Consistency)**.** *A set Γ is consistent if $\Gamma \nvdash \bot$, and inconsistent otherwise.*

Now a few basic consequences of $AX_{M1-\mathfrak{PDL}}$ can be shown. The first one is a probabilistic variant of necessitation. It captures the semantical property that deontic formula represents certain knowledge when it is a theorem of deontic logic , and therefore it must have probability value 1. The third part of the lemma shows that a form of additivity proposed as an axiom in [2] is provable in $AX_{M1-\mathfrak{PDL}}$.

Proposition 1. *The following rules are derivable form $AX_{M1-\mathfrak{PDL}}$ axiomatization:*

1. *From ϕ infer $w(\phi) = 1$*
2. $\vdash w(\bot) = 0$
3. $\vdash w(\phi \wedge \psi) + w(\phi \wedge \neg\psi) = w(\phi)$.

Proof.

1. Let us assume that a formula ϕ is derived. Then, using propositional reasoning (Taut and MP), one can infer $\phi \leftrightarrow \top$. Consequently, $w(\phi) = w(\top)$ follows from the rule P-Dis. Since we have that $w(\top) = 1$ (by W3), we can employ the axioms for reasoning about inequalities to infer $w(\phi) = 1$.
2. Then to show that $w(\bot) = 0$ using finite additivity (W2) $w(\top \vee \neg\top) = w(\top) + w(\neg\top) = 1$ and so $w(\neg\top) = 1 - w(\top)$. Since $w(\top) = 1$ and $\neg\top \leftrightarrow \bot$ we can derive $w(\bot) = 0$.

3. To derive additivity: $w(\phi) = w(\phi \wedge \psi) + w(\phi \wedge \neg\psi)$; we begin with the propositional tautology $\psi \vee \neg\psi \leftrightarrow \neg\phi \vee \neg\psi \vee \psi \leftrightarrow (\neg\phi \vee \neg\psi) \vee (\neg\phi \vee \psi) \leftrightarrow \neg(\phi \wedge \psi) \vee \neg(\phi \wedge \neg\psi) \leftrightarrow \neg((\phi \wedge \psi) \wedge (\phi \wedge \neg\psi))$ then the following equation is given by W2 $w(\phi \wedge \psi) + w(\phi \wedge \neg\psi) = w((\phi \wedge \psi) \vee (\phi \wedge \neg\psi))$. The disjunction $(\phi \wedge \psi) \vee (\phi \wedge \neg\psi)$ can be rewritten to $\phi \wedge (\psi \vee \neg\psi)$ which is equivalent to ϕ. From $\phi \leftrightarrow (\phi \wedge \psi) \vee (\phi \wedge \neg\psi)$, using P-Dis we obtain $w(\phi) = w(\phi \wedge \psi) + w(\phi \wedge \neg\psi)$.

□

Soundness and Completeness In this section we prove that the given logic construction is sound and complete with respect to the class of measurable models, combining and adapting the approaches from [2, 16].

Theorem 1 (Soundness & Completeness)**.** *The axiom system $AX_{M1-\mathfrak{PDL}}$ is sound and complete with respect to the class of measurable probabilistic deontic models. i.e., $\vdash \theta$ iff $\models \theta$.*

Proof. The proof of soundness is straightforward. As an illustration, proof of soundness of $O - K$ and $W3$ will be shown.

The axiom $O - K$ gives $O(\phi \rightarrow \phi') \rightarrow (O\phi \rightarrow O\phi')$. To show that this axiom is sound we consider some model M and some s such that $M, s \models O(\phi \rightarrow \phi')$ and $M, s \models O\phi$. Then it needs to be shown that $\forall t \in W_s$, if $w_s R_s t$, then $D_s, t \models \phi'$. So let t in D_s be such that $w_s R_s t$. Then with the two initial assumptions we get $D_s, t \models \phi \rightarrow \phi'$ and $D_s, t \models \phi$. This follows from the operator O's definition. Then by detaching $\rightarrow$ we get $D_s, t \models \phi'$ which shows the axiom is sound.

Also soundness of $W3$ can be shown; i.e. $w(\top) = 1$. Consider a model M we show that $M \models w(\top) = 1$ holds. Due to the definition of $w()$ we get $M \models w(\top) = 1$ iff $\mu(\|\top\|) = 1$. Then since every state $s \in S$ has $M, s \models \top$ it holds that $\|\top\| = S$. Then since the model is defined such that $\mu(S) = 1$ we can conclude that $w(\top) = 1$ is sound.

To prove completeness, we need to show that every consistent formula θ is satisfied in a measurable model. Since we have two types of formulas, we distinguish two cases.

If $\theta \in \mathcal{L}_{MDL}$ we write θ as ϕ. Since ϕ is consistent and monadic deontic logic is complete [8], we know that there is a MDL model (W, R, V) and $w \in W$ such that $(W, R, V), w \models \phi$. Then, for any probabilistic deontic model M with only one state s and $\tau(s) = ((W_s, R_s, V_s), w_s) = ((W, R, V), w)$ we have $M, s \models \phi$, and therefore $M \models \phi$ (since s is the only state); so the formula is satisfiable.

When $\theta \in \mathcal{L}_{M1-\mathfrak{PDL}}$ we write θ as f, and assuming consistency of f we need to prove that it is satisfiable. First notice that f can be equivalently rewritten as a formula in disjunctive normal form

$$f \leftrightarrow g_1 \vee \cdots \vee g_n$$

this means that satisfiability of f can proven by showing that one of the disjuncts g_i of the disjunctive normal form of f is satisfiable. Note that every disjunct is of the form

$$g_i = \bigwedge_{j=1}^{r} \sum_k a_{j,k} w(\phi_{j,k}) \geq c_j \wedge \bigwedge_{j=r+1}^{r+s} \neg \sum_k a_{j,k} w(\phi_{j,k}) \geq c_j$$

In order to show that g_i is satisfiable we will substitute each weight term $w(\phi_{j,k})$ by a sum of weight terms that take as arguments formulas from the set Δ that will be constructed below. For any formula ϕ, let us denote the set of subformulas of ϕ by $Sub(\phi)$. Then, for a considered formula we introduce the set of all deontic subformulas $Sub_{MDL}(\phi) = Sub(\phi) \cap \mathcal{L}_{MDL}$. We create the set Δ as the set of all possible formulas that are conjunctions of formulas from $Sub_{MDL}(g_i) \cup \{\neg e \mid e \in Sub_{MDL}(g_i)\}$, such that for every e either e or $\neg e$ is taken as a conjunct (but not both). Then we can prove the following two claims about the set Δ:

- The conjunction of any two different formulas δ_k and δ_l from Δ is inconsistent: $\vdash \neg(\delta_k \wedge \delta_l)$. This is the case because for each pair of δ's at least one subformula $e \in Sub_{MDL}(g_i)$ such that $\delta_k \wedge \delta_l \vdash e \wedge \neg e$ and $e \wedge \neg e \vdash \bot$. If there is no such e then by construction $\delta_k = \delta_l$.
- The disjunction of all δ's in Δ is a tautology: $\vdash \bigvee_{\delta \in \Delta} \delta$. This disjunction is a tautology because it contains for each subformula $e \in Sub_{MDL}(g_i)$: $e \vee \neg e$. To be more specific, there is $\forall e \in Sub_{MDL}(g_i)$ a δ_k and δ_l such that the δ's are the same except for e, thus implying $\top$. Indeed, it is clear from the way the set Δ is constructed, that the disjunction of all formulas is an instance of a propositional tautology.

As noted earlier we will substitute each term of each weight formula of g_i by a sum of weight terms. This can be done by using the just introduced set Δ and the set Φ, which we define as the set containing all deontic formulas $\phi_{j,k}$ that occur in the weight terms of g_i. In order to get all the relevant δ's to represent a weight term we construct for each $\phi \in \Phi$ the set $\Delta_\phi = \{\delta \in \Delta \mid \delta \vdash \phi\}$ which contains all δ's that imply ϕ. Than we can derive the following equivalence:

$$\vdash \phi \leftrightarrow \bigvee_{\delta \in \Delta_\phi} \delta.$$

From the rule P-Dis we obtain

$$\vdash w(\phi) = w(\bigvee_{\delta \in \Delta_\phi} \delta).$$

Then since any two elements of Δ are inconsistent we can use W2 and using axioms about inequalities we obtain

$$\vdash w(\bigvee_{\delta \in \Delta_\phi} \delta) = \sum_{\delta \in \Delta_\phi} w(\delta).$$

Consequently, we have

$$\vdash w(\phi) = \sum_{\delta \in \Delta_\phi} w(\delta).$$

Note that some of the formulas δ's might be inconsistent (for example, a formula from Δ might be a conjunction in which both Op and $F(p \wedge q)$ appear as conjuncts). For an inconsistent formula δ, we have $\vdash \delta \leftrightarrow \bot$ and consequently $\vdash w(\delta) = 0$, by the inference rule P-Dis. This can, using the axioms about linear inequalities, provably filter out the inconsistent δ's from each weight formula.

Thus, without any loss of generality, we can assume in the rest of the proof that all the formulas from Δ are consistent.

Lets us consider a new formula f', created by substituting each term of each weight formula of g_i:

$$f' = \left(\bigwedge_{j=1}^{r} \sum_k a_{j,k} \sum_{\delta \in \Delta_{\phi_{j,k}}} w(\delta) \geq c_j \right) \wedge \left(\bigwedge_{j=r+1}^{r+s} \neg \sum_k a_{j,k} \sum_{\delta \in \Delta_{\phi_{j,k}}} w(\delta) \geq c_j \right)$$

Then we will construct f'' by adding to f': a non-negativity constraint and an equality that binds the total probability weight of δ's to 1. In other words, f'' is the conjunction of the following formulas:

$$\begin{array}{lr}
 & \sum_{\delta \in \Delta} w(\delta) = 1 \\
\forall \delta \in \Delta & w(\delta) \geq 0 \\
\forall j \in \{1, \ldots, r\} & \sum_k a_{j,k} \sum_{\delta \in \Delta_{\phi_{j,k}}} w(\delta) \geq c_j \\
\forall j \in \{r+1, \ldots, r+s\} & \sum_k a_{j,k} \sum_{\delta \in \Delta_{\phi_{j,k}}} w(\delta) < c_j
\end{array}$$

Since the weights can be attributed independently while respecting the system of equations, the formula f'' is satisfiable if the following system of equations is solvable. With $I = \{1, \ldots, |\Delta|\}$ and $I_{j,k} \subseteq I$ containing the indices of the x_i that correspond with the $\delta \in \Delta_{\phi_{j,k}}$:

$$\begin{array}{lr}
 & \sum_{i=1}^{|\Delta|} x_i = 1 \\
\forall i \in I & x_i \geq 0 \\
\forall j \in \{1, \ldots, r\} & \sum_k a_{j,k} \sum_{i \in I_{j,k}} x_i \geq c_j \\
\forall j \in \{r+1, \ldots, r+s\} & \sum_k a_{j,k} \sum_{i \in I_{j,k}} x_i < c_j
\end{array}$$

Since MDL is sound and complete there is for each consistent δ there is a pointed MDL model that satisfies it. We choose the set of states S to consist of those formulas δ, and each such state is connected to the corresponding pointed MDL model via the identification function τ. The set $\mathscr{X}$ is the power-set of S which leaves setting the probability measure μ to a solution of the system given above. This means that to prove satisfiability only a solution should be found that corresponds with the system of equations above. Due to adding the constraints of probability measures to the representation, we can find a probability measure by solving the system of linear inequalities f'', using the axioms for reasoning with inequalities I1-I7. We took f in the beginning of the proof to be a consistent formula and f is either satisfiable or unsatisfiable. When the system can be shown to be satisfiable we have proven completeness, satisfiability of f is proven when a solution to f'' is shown to exist. This is the case because if f'' is solvable then so is f' which means g_i is satisfiable and if g_i is satisfiable then f is satisfiable. Assume no solution to f'' exists then $\neg f''$ is provable from the axioms I1-I7. As just explained f's satisfiability is equivalent to existence of a solution to f''. Therefore $\neg f$ is provable which means that f is inconsistent. This is a contradiction, and therefore we have to reject the assumption that f'' is unsatisfiable; so we conclude that f is satisfiable. $\square$

3.2 Type 2 Monadic $\mathfrak{PDL}$

In this section a second monadic model is developed, instead of using a type 1 probability structure this construction uses a type 2 probability structure. To do this the language is changed since in this type of probability structure weight formulas can occur inside weight formulas. The axiomatic system and satisfaction relation change also due to indexation and the difference in language. Lastly the model definition changes relatively more rigorously due to the change in probability structure.

The modified logic will have a probability structure that gives a probability measure to each agent-state (i, s) pair: $(S_{i,s}, \mathscr{X}_{i,s}, \mu_{i,s})$. The probability structure is extended in such a way that the logic allows formulas relating to Theory of Mind. For example $w_j(w_i(\phi) \geq \alpha) \geq \beta$ which depicts uncertainty of agent j about agent i's uncertainty about ϕ, this means that a weight function $w_k()$ includes next to deontic formulas also the weight formulas of other agents in its domain.

Syntax and Semantics

Definition 25 (Formulae)**.** *Let $\mathbb{P}$ be a set of atomic propositions. The language $\mathcal{L}_{M2-\mathfrak{PDL}}$ of type 2 monadic probabilistic deontic logic is generated by the following two sentences of BNF (Backus Naur Form):*

$$
\begin{array}{lll}
[\mathcal{L}_{MDL}] & \phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid O\phi & p \in \mathbb{P} \\
[\mathcal{L}_{M2-\mathfrak{PDL}}] & \theta ::= \phi \mid a_1 w_i(\theta_1) + \cdots + a_n w_i(\theta_n) \geq \alpha \mid \neg\theta \mid \theta \wedge \theta &
\end{array}
$$

The construct $O\phi$ is read as "It is obligatory that ϕ", and $w_i()$ in the construct $M, s \models w_i(\phi) \geq \alpha$ stands for "weight according to agent i" and depicts the probability (certainty/uncertainty) with respect to a deontic formula ϕ, agent i and state s in model M. Both languages also give rise to other connectives as in definition 17.

Example 2. *Following the earlier example about protecting life, the fact that a rules is considered established might be expressed by the probabilistic statement "the probability that one is obliged to protect life is at least 0.9". This sentence could be formalized using the new probability structure as*

$$w_i(Oq) \geq 0.9.$$

where the individual referred to as one can be identified with i.

Definition 26 (Model)**.** *A type 2 monadic probabilistic deontic model is a tuple $M = \langle S, \tau, \mathscr{P} \rangle$, where:*

- *S is a non-empty set of states*
- *τ associates with each state $s \in S$ a tuple containing a monadic deontic model and one of its worlds: $\tau(s) = (D_s, w_s)$ where:*
  - *$D_s = (W_s, R_s, V_s)$ a monadic deontic model*
  - *$w_s \in W_s$ is a world of the monadic deontic model*
- *$\mathscr{P}(i, s) = (S_{i,s}, \mathscr{X}_{i,s}, \mu_{i,s})$ is a function assigning to each combination of agent (i) and state (s) a probability space where:*
  - *$S_{i,s} \subseteq S$ an arbitrary subset of S that can be interpreted as the set of states that agent i has conceptions about in state s.*
  - *$\mathscr{X}_{i,s}$ is a σ-algebra of subsets of $S_{i,s}$*
  - *$\mu_{i,s} : \mathscr{X}_{i,s} \mapsto [0, 1]$ is a standard probability measure as defined in Definition 10.*

Let us illustrate this definition.

Example 2. ***(continued)***
This model type is depicted in Figure 2. The circle on the right contains the four states of the model, which are measured by probability measures $\mu_{i,s}$. Each of the state-agent pairs is equipped with a standard probability space with as domain a pointed model of MDL. In this picture the dotted lines represent the measure $\mu_{i,s}$ for which each edges originates from s and goes to a state in $S_{i,s}$. Note that the arrows depict the "good" alternative relation R. If we assume that q stands for "protect life", like in the previous example, in all good successors of w_1 the proposition q holds.

Assume a finite set of atomic propositions $\{p, q\}$. Note we define only the probabilities attributed by an agent i evaluated at state s. Because of this reason four probabilities are defined. Let us consider the model $M = \langle S, \tau, \mathscr{P} \rangle$, where

- *$S = \{s, s', s'', s'''\}$*
- *$\mathscr{P}(i, s)$*
  - *$S_{i,s}$ an arbitrary subset of S in this case S.*
  - *$\mathscr{X}_{i,s}$ is the set of all subsets of $S_{i,s}$*
  - *$\mu_{i,s}$ is characterized by: $\mu_{i,s}(\{s\}) = 0.5$, $\mu_{i,s}(\{s'\}) = \mu_{i,s}(\{s''\}) = 0.2$, $\mu_{i,s}(\{s'''\}) = 0.1$ (other values follow from the properties of probability measures)*
- *τ is a mapping which assigns to the state s, $D_s = (W_s, R_s, V_s)$ and w_s such that*
  - *$W_s = \{w_1, w_2, w_3, w_4\}$*
  - *$R_s = \{(w_1, w_2), (w_1, w_3), (w_2, w_2), (w_2, w_3), (w_3, w_2), (w_3, w_3), (w_4, w_2), (w_4, w_3), (w_4, w_4)\}$*
  - *$V_s(p) = \{w_1, w_3\}$, $V_s(q) = \{w_2, w_3\}$*
  - *$w_s = w_1$*

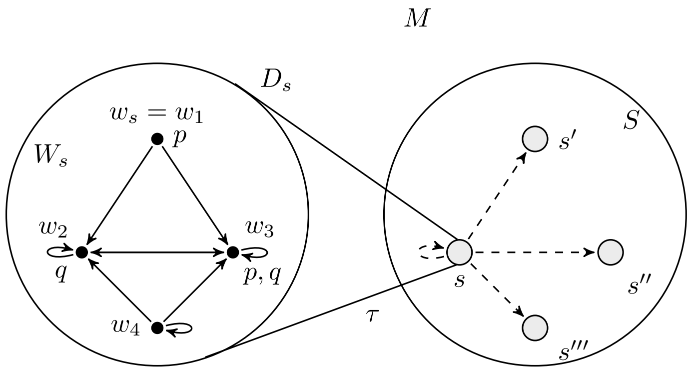

Figure 2: Model $M = \langle S, \tau, \mathscr{P} \rangle$.

Note that the domain of τ is always the whole set S, but in this example we only explicitly specify $\tau(s)$ for illustration purposes.

Next the satisfiability of a formula in a model can be defined. First the truth of a deontic formula in a state of a type 2 monadic probabilistic deontic model is given. This definition is in accordance with the standard satisfiability relation of MDL $\models_{MDL}$.

Definition 27 (Satisfaction)**.** *Let $M = \langle S, \tau, \mathscr{P} \rangle$ be a $M2 - \mathfrak{PDL}$ model, and let $s \in S$. We define the satisfiability of formula $\theta \in \mathcal{L}_{M2-\mathfrak{PDL}}$, in state s of model M denoted by $M, s \models \theta$ recursively as follows with $\phi \in \mathcal{L}_{MDL}$:*

- $M, s \models \phi$ *iff* $(D_s, w_s) \models_{MDL} \phi$.
- $M, s \models a_1 w_i(\theta_1) + \cdots + a_n w_i(\theta_n) \geq \alpha$ *iff* $a_1 \mu_{i,s}(\|\theta_1\|^M_{i,s}) + \cdots + a_n \mu_{i,s}(\|\theta_n\|^M_{i,s}) \geq \alpha$.
- $M, s \models \neg\theta$ *iff* $M, s \not\models \theta$.
- $M, s \models \theta_l \wedge \theta_k$ *iff* $M, s \models \theta_l$ *and* $M, s \models \theta_k$.

For a model $M = \langle S, \tau, \mathscr{P} \rangle$, a formula $\theta \in \mathcal{L}_{M2-\mathfrak{PDL}}$, state s and agent i, let $\|\theta\|^M_{i,s}$ denote the set of states that satisfy θ, from the perspective of agent i in state s i.e.,

$$\|\theta\|^M_{i,s} = \{s' \in S_{i,s} \mid M, s' \models \theta\}.$$

We omit the super- and subscripts from $\|\theta\|^M_{i,s}$ when it is clear from context. The following definition introduces an important class of probabilistic deontic models, so called measurable *models. The satisfaction relation shows that in this model construction formulas θ can occur as the argument to a weight formula w, this means that weight formulas can be arguments of weight operators.*

Since focus is on measurable structures and completeness is proven for this class of structures, this class is redefined for type 2 models.

Definition 28 (Measurable model)**.** *A probabilistic deontic model is* measurable *if*

$$\|\phi\|_{i,s}^{M} \in \mathscr{X}_{i,s}$$

for every $\phi \in \mathcal{L}_{MDL}$.

Example 2. ***(continued)*** *Continuing the previous example, according to Definition 27 it holds that $M, s \models Oq$. At this point it is also possible to speak of the uncertainty of agent i of the obligation to protect life. Assume that τ is defined in the way such that $M, s' \models Oq$ and $M, s'' \models Oq$, but $M, s''' \not\models Oq$. Then $\mu_{i,s}(\|Oq\|) = \mu_{i,s}(\{s, s', s''\}) = 0.5 + 0.2 + 0.2 = 0.9$. According to Definition 19, $M, s \models w_i(Oq) \geq 0.9$. This can be extended in a similar way to describe the uncertainty of agent j about the uncertainty of agent i of the obligation to protect life.*

Axiomatization The following axiomatization $AX_{M2-\mathfrak{PDL}}$ combines the axioms of proof system D of monadic deontic logic [8] with the axioms of the type 2 probabilistic logic [1]. The axioms for reasoning about linear inequalities are taken form [2].

Tautologies and Modus Ponens

Taut. All instances of propositional tautologies.

MP. From θ and $\theta \rightarrow \theta'$ infer θ'.

Reasoning with O:

O-... see axiomatization in Section 3.1

Reasoning about linear inequalities:

I1.-I7. see axiomatization in Section 3.1

Reasoning about probabilities:

W1. $w_i(\theta) \geq 0$ (non negativity).

W2. $w_i(\theta \vee \theta') = w_i(\theta) + w_i(\theta')$, if $\neg(\theta' \wedge \theta')$ is an instance of a classical propositional tautology (finite additivity).

W3. $w_i(\top) = 1$

P-Dis. From $\theta \leftrightarrow \theta'$ infer $w_i(\theta) = w_i(\theta')$ (probabilistic distributivity)

The axiom Taut allows all of propositional tautologies. Note that Modus Ponens (MP) can be applied to both types of formulas, since $\mathcal{L}_{MDL}$ is included in $\mathcal{L}_{M2-\mathfrak{PDL}}$. O-Nec is a deontic variant of the necessitation rule. P-Dis is an inference rule which states that two equivalent deontic formulas must have the same probability values.

Soundness and Completeness In this section it is proven that the construction $M2 - \mathfrak{pDL}$ is sound and complete with respect to the class of measurable models, combining and adapting the approaches from [2, 16, 1].

Theorem 2 (Soundness & Completeness)**.** *The axiom system $AX_{M2-\mathfrak{pDL}}$ is sound and complete with respect to the class of measurable probabilistic deontic models. i.e., $\vdash \theta$ iff $\models \theta$.*

Proof. The proof is a modification of the proof for M1-$\mathfrak{pDL}$and is based on the proof of soundness and completeness in [1] and [2]. To prove completeness, we need to show that every consistent formula θ is satisfiable in a measurable model. The modification of the logic gives iterations of weight formulas of arbitrary depth, also instead of one measure there is a measure for each agent and state pair (i, s); for this the proof needs to be adjusted.

For any formula ψ we will denote the set of subformulas closed under negation as follows $Sub^+(\psi) = Sub(\psi) \cup \{\neg e \mid e \in Sub(\psi)\}$. And we call a set A a maximal consistent set with respect to a set B when A is maximal and consistent with respect to B. Consistency is defined in Definition 24 and A is maximal with regards to B when $\forall e \in B$, A contains either e or $\neg e$.

Let θ be a consistent formula of $\mathcal{L}_{M2-\mathfrak{pDL}}$. Then let S denote the set of maximal consistent subsets of $Sub^+(\theta)$. And define for each $s \in S$ the element $\xi_s = \bigwedge_{\epsilon \in s} \epsilon$ to be the conjunction of elements in s. Denote the set of elements ξ_s as follows $\Xi = \{\xi_s \mid s \in S\}$. Each of the conjunctions $\xi_s \in \Xi$ is identified with a set s. These sets can then be identified as a state, and S is the set of states in our model. Furthermore, we define τ in the following way. By soundness and completeness of MDL, for each ξ_s there is a deontic model D_s and a world w_s in it such that $D_s, w_s \models_{MDL} \xi_s$. Then we define $\tau(s) = (D_s, w_s)$.

Since our probabilistic deontic model is of the form $M = (S, \tau, \mathscr{P})$ this leaves the task of defining $\mathscr{P}$ the probability assignment. In such a way that when we consider the model M, then for every $s \in S$ and every formula $\psi \in Sub^+(\theta)$ we have $M, s \models \psi$ iff $\psi \in s$. To do this we will make use of additivity using the following equivalence:

$$\vdash \psi \leftrightarrow \bigvee_{\{s \in S \mid \psi \in s\}} \xi_s$$

where ξ_s is the $\xi \in \Xi$ that corresponds with state s. Then this leads to the following equation:

$$\mu_i(\psi) = \sum_{\{s \in S \mid \psi \in s\}} \mu_i(\xi_s)$$

Using this together with I2-I4, it can be shown that a probability formula $\psi \in Sub^+(\theta)$ is provably equivalent to a formula of the form $\sum_{s \in S} c_s \mu_i(s) \geq b$, for some appropriate coefficients c_s.

Using this derivation and by setting an agent i and a state $s \in S$. It is possible to describe a set of linear equalities and inequalities corresponding to i, s and θ, over variables of the form $x_{iss'}$, for $s' \in S$. We can think of $x_{iss'}$ as representing $\mu_{i,s}(s')$, that is the probability of state s' under agent i's probability distribution at state s. For each probability formula $\psi \in Sub^+(\theta)$ we will have a corresponding inequality. Since ψ is a probability formula it is equivalent to $\sum_{s' \in S_{i,s}} c_{s'} \mu_i(s') \geq \beta$ with $c_{s'}, \beta \in \mathbb{Q}$. As in the proof in Theorem 1 we can observe that by construction each state s contains exactly one of either ψ and $\neg\psi$. When $\psi \in s$ then the corresponding inequality is

$$\sum_{s' \in S} c_{s'} x_{iss'} \geq \beta.$$

When $\neg\psi \in s$, then the corresponding inequality is

$$\sum_{s' \in S} c_{s'} x_{iss'} < \beta.$$

This inequality can be constructed for each $\psi \in Sub^+(\theta)$. Then the following equation constrains the total probability mass to 1 for the measure in state s belonging to agent i.

$$\sum_{s' \in S} x_{iss'} = 1.$$

Finally, this equation gives non-negativity

$$\forall s' \in S \quad x_{iss'} \geq 0$$

As shown in [2] this system of inequalities has a solution $x^*_{iss'}$ for all $s' \in S$; since each ξ_s is consistent.

We build the model by calling for soundness and completeness of MDL. This means that for each χ there is a pointed MDL model in which it is satisfied. Then for each χ we create an $s \in S$ such that it is connected via τ to the pointed MDL model in which χ is satisfied. The set $\mathscr{X}$ is automatically generated from its respective subset of S. The for each i and each s, we solve the corresponding set of inequalities separately to determine the respective probability measure. Then $\mathscr{P}$ can be defined such that $\mathscr{P}(i, s) = (S, 2^S, \mu_{i,s})$, where if $A \subseteq S$, then $\mu_{i,s}(A) = \sum_{s' \in A} x^*_{iss'}$. Since $\sum_{s' \in S} x^*_{iss'} = 1$, it is easy to see that $\mu_{i,s}$ is indeed a probability measure. Note that, in the probability space $\mathscr{P}(i, s)$, every set is measurable.

As said above, what is left is to show that for every formula $\psi \in Sub^+(\theta)$ and every state in S, we have $M, s \models \psi$ iff $\psi \in s$. The proof proceeds by induction on ψ. If ψ is a deontic formula the result is immediate from the definition of τ. The cases where ψ is a negation or a conjunction are straightforward. The case where ψ is an i-probability formula follows immediately from the arguments above, since the appropriate inequality corresponding to ψ is satisfied by $\mu_{i,s}$. This means that if the formula θ is consistent then it must occur in one of the maximal consistent subsets of $Sub^+(\theta)$ therefore θ is satisfiable in a model, and so the logic is complete. $\square$

3.3 Type 1 Dyadic $\mathfrak{PDL}$

In this subsection the definitions are given that change when we instead of using MDL use DDL in conjunction with the type 1 probability structure of [2] that was introduced in 3.1. Remember that DDL can be viewed as a generalization of MDL. The main difference is that DDL uses a dyadic (conditional) Deontic operator $O(\phi|\psi)$ which means that ϕ is obligatory when ψ holds. This allows the ranking of states based on the number of obligations that are not met in each of the states. This logic will therefore allow formulas that describe uncertainty with regards to dyadic obligations. This depicts the prevalence of states in which the dyadic obligation is met.

Syntax and Semantics

Definition 29 (Formulae). *Let $\mathbb{P}$ be a set of atomic propositions. The language $\mathcal{L}_{D1-\mathfrak{PDL}}$ of type 1 dyadic probabilistic deontic logic is generated by the following two sentences of BNF (Backus Naur Form):*

$$[\mathcal{L}_{DDL}] \qquad \phi ::= p \mid \neg\phi \mid (\phi \wedge \phi) \mid \Box\phi \mid \Diamond\phi \mid O(\phi|\phi) \qquad p \in \mathbb{P}$$
$$[\mathcal{L}_{D1-\mathfrak{PDL}}] \qquad f ::= a_1 w(\phi_1) + \cdots + a_n w(\phi_n) \geq \alpha \mid \neg f \mid (f \wedge f)$$

The construct $\Box\phi$ is read as "ϕ is settled as true", and $O(\psi|\phi)$ as "ψ is obligatory, given ϕ". ϕ is called the antecedent, and ψ the consequent. $P(\psi|\phi)$ ("ψ is permitted, given ϕ") is short for $\neg O(\neg\psi|\phi)$, $O\phi$ ("ϕ is unconditionally obligatory") and $P\phi$ ("ϕ is unconditionally permitted") are short for $O(\phi|\top)$ and $P(\phi|\top)$, respectively. $\Diamond\phi$ is short for $\neg\Box\neg\phi$. And $w()$ in the construct $w(\phi) \geq \alpha$ stands for "weight" and depicts the probability (certainty/uncertainty) with respect to a deontic formula ϕ. Both languages also give rise to other connectives as in definition 17.

Example 3. *Following the example used previously, the fact that a rule is considered established can be expressed by the probabilistic statement "the probability that one is obliged to protect life is at least 0.4". This sentence could be formalized using the introduced language as*

$$w(Oq \mid \top) \geq 0.4.$$

Definition 30 (Model). *A type 1 dyadic probabilistic deontic model is a tuple $M = \langle S, \mathscr{X}, \mu, \tau \rangle$, which combines dyadic deontic logic models with a type 1 standard probability measure where:*

- *S is a non-empty set of* states
- *$\mathscr{X}$ is a σ-algebra of subsets of S*
- *$\mu : \mathscr{X} \to [0,1]$ is a standard probability measure as defined in Definition 10.*
- *τ associates with each state in S a tuple containing a dyadic deontic model and one of its worlds: $\tau(s) \mapsto (D_s, w_s)$ where:*
  - *$D_s = (W_s, \succeq_s, V_s)$ is a model of dyadic deontic logic.*
  - *$w_s \in W_s$ is a world w_s in W_s of model D_s.*

Example 3. ***(continued)*** *A model of this type is depicted in Figure 3. The circle on the right contains the four states of the model, which are measured by μ. Each of the states is equipped with a standard pointed model of DDL. In this picture, only one of them is shown, the one that corresponds to s. It ($\tau(s)$) is represented within the circle on the left. Note that the alternative relation R is substituted for a preference relation $\succeq_s$. If we assume that q stands for "protect life", like in the previous example, in all best successors of w_1 the proposition q holds. Note that, according to Definition 6, this means that in w_1 proposition q is considered obligatory ie. $O(q|\top)$.*

Assume a finite set of atomic propositions $\{p, q\}$. Let us consider the model $M = \langle S, \mathscr{X}, \mu, \tau \rangle$, where

- $S = \{s, s', s'', s'''\}$
- $\mathscr{X}$ *is the set of all subsets of* S
- μ *is characterized by:* $\mu(\{s\}) = 0.5$, $\mu(\{s'\}) = \mu(\{s''\}) = 0.2$, $\mu(\{s'''\}) = 0.1$ *(other values follow from the properties of probability measures)*
- τ *is a mapping which assigns to the state* s, $D_s = (W_s, R_s, V_s)$ *and* w_s *such that*
  - $W_s = \{w_1, w_2, w_3, w_4\}$
  - $opt_{\succeq_s}(\|\top\|) = \{w2, w3\}$
  - $V_s(p) = \{w_1, w_3\}$, $V_s(q) = \{w_2, w_3\}$
  - $w_s = w_1$

  Note that the domain of τ *is always the whole set* S*, but in this example we only explicitly specify* $\tau(s)$ *for illustration purposes.*

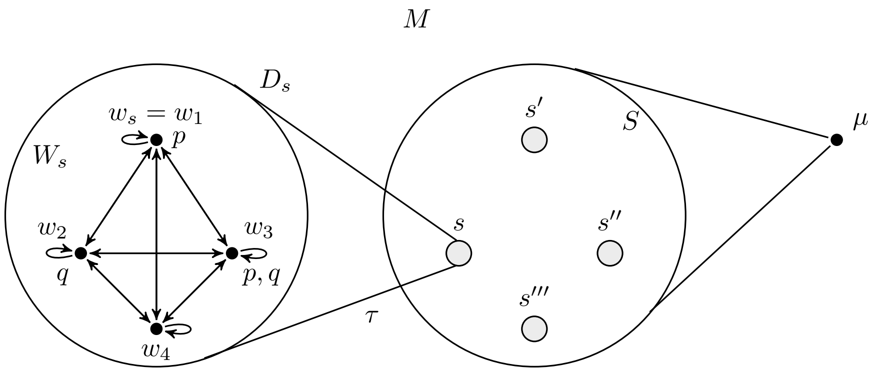

Figure 3: Model $M = \langle S, \mathscr{X}, \mu, \tau \rangle$.

Definition 31 (Satisfaction)**.** *Let* $M = \langle S, \mathscr{X}, \mu, \tau \rangle$ *be a measurable type 1 dyadic probabilistic deontic model. We define the satisfiability relation* $\models$ *recursively as follows using* $\models_{DDL}$*:*

- $M \models \phi$ *iff* $\forall s \in S$, $D_s, w_s \models_{DDL} \phi$
- $M \models a_1 w(\phi_1) + \cdots + a_k w(\phi_k) \geq \alpha$ *iff* $a_1\mu(\|\phi_1\|) + \cdots + a_k\mu(\|\phi_k\|) \geq \alpha$.
- $M \models \neg f$ *iff* $M \not\models f$
- $M \models f \wedge g$ *iff* $M \models f$ *and* $M \models g$

Where $\|\phi\|$ *has the same meaning as before namely the set of states that satisfy* ϕ*. As shown the only difference for satisfaction relations compared to* $M1 - \mathfrak{PDL}$ *is the change from* $\models_{MDL}$ *to* $\models_{DDL}$*. As before, we restrict our attention to the class of measurable models, where we say that a model is measurable if* $\|\phi\|^M \in \mathscr{X}$ *for every* $\phi \in \mathcal{L}_{DDL}$*.*

Example 3. ***(continued)*** *Continuing the previous example, according to Definition 31 it holds that $M, s \not\models O(q \mid \top)$. At this point it is also possible to speak of the probability of the obligation to protect life. Assume that τ is defined in the way such that $M, s' \models O(q \mid \top)$ and $M, s'' \models O(q \mid \top)$, but $M, s''' \not\models O(q \mid \top)$. Then $\mu(\|Oq\|) = \mu(\{s, s', s''\}) = 0.2 + 0.2 = 0.4$. According to Definition 31, $M, s \models w(O(q \mid \top) \geq 0.4$.*

Axiomatization This axiomatization call it $AX_{D1-\mathfrak{PDL}}$ is similar to $AX_{M1-\mathfrak{PDL}}$. The MDL axioms have been exchanged for DDL axioms. The axioms system of DDL that is used uses a lot more axioms than the axiom system of MDL, it uses 10 axioms compared to the previously 3 axioms. This is also due to the inclusion of $\Box$ into the logic. For reasoning about probabilities we use the same axioms as used for reasoning about probabilities in $AX_{M1-\mathfrak{PDL}}$.

Tautologies and Modus Ponens

Taut. All instances of propositional tautologies.

MP. From θ and $\theta \rightarrow \theta'$ infer θ'.

Reasoning with $\Box$:

$\Box$-K. $\Box(\phi \rightarrow \phi') \rightarrow (\Box\phi \rightarrow \Box\phi')$

$\Box$-T. $\Box\phi \rightarrow \phi$

$\Box$-5. $\neg\Box\phi \rightarrow \Box\neg\Box\phi$

$\Box$-Nec. If $\vdash \phi$ then $\vdash \Box\phi$

Reasoning with $O(-|-)$:

COK. $O(\phi' \rightarrow \phi''|\phi) \rightarrow (O(\phi'|\phi) \rightarrow O(\phi''|\phi))$

id. $O(\phi|\phi)$

Sh. $O(\phi''|(\phi \wedge \phi')) \rightarrow O((\phi' \rightarrow \phi'')|\phi)$

Interplay of $\Box$ and $O(-|-)$:

Abs. $O(\phi'|\phi) \rightarrow \Box O(\phi'|\phi)$

Nec. $\Box\phi' \rightarrow O(\phi'|\phi)$

Ext. $\Box(\phi \leftrightarrow \phi') \rightarrow (O(\phi''|\phi) \leftrightarrow O(\phi''|\phi'))$

Reasoning about linear inequalities:

I1.-I7 see axiomatization in Section 3.1

Reasoning about probabilities:

... see axiomatization in Section 3.1

Soundness and Completeness Showing soundness and completeness for $D1 - \mathfrak{PDL}$ relates to the question of whether the proof of soundness and completeness of $M1 - \mathfrak{PDL}$ can be used. The probability structure of these logics is both the type 1 probability structure. The difference is in deontic logic, $D1 - \mathfrak{PDL}$ uses Dyadic Deontic Logic and $M1 - \mathfrak{PDL}$ uses Monadic Deontic Logic. When we look at Definition 31 we can see that satisfiability depends on $\models_{DDL}$. In a similar way we can see that in Definition 19 satisfiability in $M1 - \mathfrak{PDL}$ depends on $\models_{MDL}$. This shows that the change from MDL to DDL affects the proof of Theorem 1 in relation to the construction of τ. To compensate for this instead of calling for soundness and completeness of MDL models we do so for DDL models.

3.4 Type 2 Dyadic $\mathfrak{PDL}$

In this subsection the definitions are given that change when we instead of using MDL use DDL in conjunction with the type 2 probability structure of [1] that was introduced in section 3.2.

Syntax and Semantics

Definition 32 (Formulae). *Let $\mathbb{P}$ be a set of atomic propositions. The language $\mathcal{L}_{D2-\mathfrak{PDL}}$ of type 2 dyadic probabilistic deontic logic is generated by the following two sentences of BNF (Backus Naur Form):*

$$\begin{array}{lll} [\mathcal{L}_{DDL}] & \phi ::= p \mid \neg\phi \mid (\phi \wedge \phi) \mid \Box\phi \mid \Diamond\phi \mid O(\phi|\phi) & p \in \mathbb{P} \\ [\mathcal{L}_{D2-\mathfrak{PDL}}] & \theta ::= \phi \mid a_1 w_i(\theta_1) + \cdots + a_n w_i(\theta_n) \geq \alpha \mid \neg\theta \mid \theta \wedge \theta & \end{array}$$

The construct $\Box\phi$ is read as "ϕ is settled as true", and $O(\psi|\phi)$ as "ψ is obligatory, given ϕ". ϕ is called the antecedent, and ψ the consequent. $P(\psi|\phi)$ ("ψ is permitted, given ϕ") is short for $\neg O(\neg\psi|\phi)$, $O\phi$ ("ϕ is unconditionally obligatory") and $P\phi$ ("ϕ is unconditionally permitted") are short for $O(\phi|\top)$ and $P(\phi|\top)$, respectively. $\Diamond\phi$ is short for $\neg\Box\neg\phi$. And $w()$ in the construct $w(\phi) \geq \alpha$ stands for "weight" and depicts the probability (certainty/uncertainty) with respect to a deontic formula ϕ. Both languages also give rise to other connectives as in definition 17.

Example 4. *Following the earlier example about protecting life, the fact that a rules is considered established might be expressed by the probabilistic statement "the probability that one is obliged to protect life is at least 0.4". This sentence could be formalized using the type 2 probability structure as*

$$w_i(O(q \mid \top) \geq 0.4.$$

where the individual referred to as one can be identified with i.

Definition 33 (Model). *A dyadic probabilistic deontic model with a type 2 probability structure is a tuple $M = \langle S, \tau, \mathscr{P} \rangle$, where:*

- *S is a non-empty set of states*
- *τ associates with each state $s \in S$ a tuple containing a dyadic deontic model and one of its worlds: $\tau(s) = (D_s, w_s)$ where:*

  - $D_s = (W_s, \succeq_s, V_s)$ *a dyadic deontic model*
  - $w_s \in W_s$ *is a world of the dyadic deontic model*

- $\mathscr{P}(i, s) = (S_{i,s}, \mathscr{X}_{i,s}, \mu_{i,s})$ *is a function assigning to each combination of agent* (i) *and state* (s) *a probability space where:*
  - $S_{i,s} \subseteq S$ *an arbitrary subset of* S *that can be interpreted as the set of states that agent* i *has conceptions about in state* s*.*
  - $\mathscr{X}_{i,s}$ *is a* σ*-algebra of subsets of* $S_{i,s}$
  - $\mu_{i,s} : \mathscr{X}_{i,s} \mapsto [0, 1]$ *is a standard probability measure as defined in Definition 10.*

Example 4. ***(continued)*** *This model type is depicted in Figure 2.*

This model construction is depicted in Figure 4. The circle on the right contains the four states of the model, which are measured by probability measures $\mu_{i,s}$*. Each of the state-agent pairs is equipped with a standard probability space with as domain a pointed model of MDL. In this picture the dotted lines represent the measure* $\mu_{i,s}$ *for which each edges originates from* s *and goes to a state in* $S_{i,s}$*. Note that the arrows depict the comparison relation* $\succeq$*. If we assume that* q *stands for "protect life", like in the previous example, then* $O(q \mid \top)$ *holds if* w_2 *and* w_3 *are considered "best".*

Assume a finite set of atomic propositions $\{p, q\}$*. Note we define only the probabilities attributed by an agent* i *evaluated at state* s*. Because of this reason four probabilities are defined. Let us consider the model* $M = \langle S, \tau, \mathscr{P}\rangle$*, where*

- $S = \{s, s', s'', s'''\}$
- $\mathscr{P}(i, s)$
  - $S_{i,s}$ *an arbitrary subset of* S *in this case* S*.*
  - $\mathscr{X}_{i,s}$ *is the set of all subsets of* $S_{i,s}$
  - $\mu_{i,s}$ *is characterized by:* $\mu_{i,s}(\{s\}) = 0.5$*,* $\mu_{i,s}(\{s'\}) = \mu_{i,s}(\{s''\}) = 0.2$*,* $\mu_{i,s}(\{s'''\}) = 0.1$ *(other values follow from the properties of probability measures)*
- τ *is a mapping which assigns to the state* s*,* $D_s = (W_s, R_s, V_s)$ *and* w_s *such that*
  - $W_s = \{w_1, w_2, w_3, w_4\}$
  - $opt_{\succeq_s}(\|\top\|) = \{w2, w3\}$
  - $V_s(p) = \{w_1, w_3\}$*,* $V_s(q) = \{w_2, w_3\}$
  - $w_s = w_1$

  Note that the domain of τ *is always the whole set* S*, but in this example we only explicitly specify* $\tau(s)$ *for illustration purposes.*

Definition 34 (Satisfaction)**.** *Let* $M = \langle S, \tau, \mathscr{P}\rangle$ *be a dyadic probabilistic deontic model, and let* $s \in S$*. We define the satisfiability of formula* $\theta \in \mathcal{L}$*, in state* s *of model* M *denoted by* $M, s \models \theta$ *recursively as follows with* $\phi \in \mathcal{L}_{DDL}$ *and* $\models_{DDL}$ *as before:*

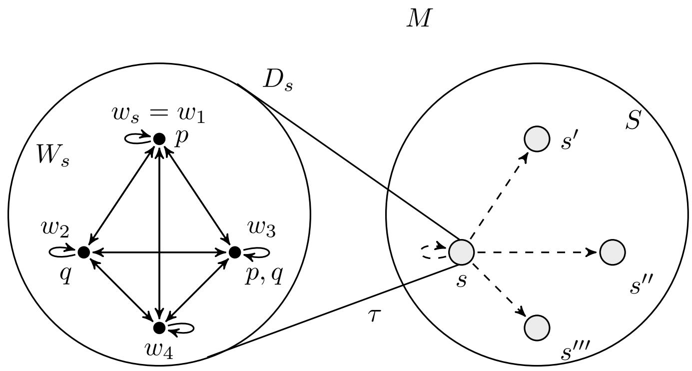

Figure 4: Model $M = \langle S, \tau, \mathscr{P} \rangle$.

- *$M, s \models \phi$ iff $D_s, w_s \models_{DDL} \phi$.*
- *$M, s \models a_1 w_i(\theta_1) + \cdots + a_n w_i(\theta_n) \geq \alpha$ iff $a_1 \mu_{i,s}(\|\theta_1\|^M_{i,s}) + \cdots + a_n \mu_{i,s}(\|\theta_n\|^M_{i,s}) \geq \alpha$.*
- *$M, s \models \neg\theta$ iff $M, s \not\models \theta$.*
- *$M, s \models \theta_l \wedge \theta_k$ iff $M, s \models \theta_l$ and $M, s \models \theta_k$.*

Where $\|\phi\|_{i,s}$ has the same meaning as before namely the set of states that satisfy ϕ , but in this case is also dependent on state and agent.

Example 4. ***(continued)*** *Continuing the previous example, according to Definition 19 it holds that $M, s \not\models O(q \mid \top)$. At this point it is also possible to speak of the probability of the obligation to protect life. Assume that τ is defined in the way such that $M, s' \models O(q \mid \top)$ and $M, s'' \models O(q \mid \top)$, but $M, s''' \not\models O(q \mid \top)$. Then $\mu_{i,s}(\|Oq\|) = \mu_{i,s}(\{s, s', s''\}) = 0.2 + 0.2 = 0.4$. According to Definition 19, $M, s \models w_i(O(q \mid \top)) \geq 0.4$: agent i considers in state s the probability of the obligation to protect life to be greater than* 0.4.

Axiomatization Let us call this axiomatization $AX_{D2-\mathfrak{PDL}}$ in a similar manner as the previous axiomatizations. All axioms are left out since they have already been introduced. This axiom systems combines the axioms from DDL with the axioms that are needed when using a type 2 probability structure. This results in a total of 23 axioms.

Tautologies and Modus Ponens

Taut. All instances of propositional tautologies.

MP. From θ and $\theta \to \theta'$ infer θ'.

Reasoning with $\Box$:

… see axiomatization in Section 3.3

Reasoning with $O(-|-)$:

... see axiomatization in Section 3.3

Interplay of $\Box$ and $O(-|-)$:

... see axiomatization in Section 3.3

Reasoning about linear inequalities:

I1.-I7 see axiomatization in Section 3.1

Reasoning about probabilities:

... see axiomatization in Section 3.2

Soundness and Completeness As with $D1-\mathfrak{PDL}$ the soundness and completeness of this new construction $D2-\mathfrak{PDL}$ is derived from soundness and completeness of $M2-\mathfrak{PDL}$. Just as when showing soundness and completeness for $D2-\mathfrak{PDL}$ the proof changes in relation to the construction of τ. Therefore instead of calling for soundness and completeness of MDL models we do so for DDL models.

4 Conclusion & Discussion

To round off the project a few things will be discussed and concluded among which possible further research but also the decisions and considerations involving the construction of the models. But first a summary of the project. In this project a new logic was constructed based on Deontic Logic and Probabilistic Logic: $\mathfrak{PDL}$. Four variants of $\mathfrak{PDL}$were constructed all based on the same construction technique but using different variants of Deontic Logic and Probabilistic Logic. The construction technique used evolves around a function τ that links Deontic worlds to Probabilistic states. This allows the formulas to express the central to be formalized notion of this project namely: uncertainty about obligation. Furthermore this construction method restricts combinations and nesting of operators O and w in such a way that the notion: Obligatory uncertainty, is excluded from the language. The first variant of $\mathfrak{PDL}$combined Monadic Deontic Logic with type 1 Probabilistic Logic formalizing the notion of uncertainty about obligations. The second variant used type 2 Probabilistic Logic and in doing so allowed uncertainty about someone else's uncertainty about obligations. Then the other two variants generalized the previous two variants by using Dyadic Deontic Logic instead of Monadic Deontic Logic. This change allowed uncertainty about conditional obligations.

Possible further research on this logic could involve defining multiplication and division on the weight terms. Explicitly including this in the logic is neat, also it is feasible since it is implemented in [2]. In [2] conditional probabilities are implemented, these follow from multiplication and division and could also be included in the framework. Another interesting modification is to change the standard probability measure into a non-standard probability measure, this is not trivial and is worthwhile with respect to expressiveness. In a more practical sense $M1-\mathfrak{PDL}$ could be used in such a way that a $M1-\mathfrak{PDL}$ model is created for each agent in a group of agents thus extending it for the multi-agent case. Furthermore the learning process could be made tangible by implementing it with for example Bayesian updating behind it. Lastly Machine Learning might be applied to the solution space of a system of inequalities. This could lead to clusters of solutions. On a side note MDL and probabilistic logic seem very compatible with preferential reasoning as in [7] looking into other correlations between the logics can be of value.

The deontic operator O and the weight function $w()$ have been combined into formulas. Two combinations between O and $w()$ are syntactically possible namely: $O(w(\phi) \geq \alpha)$ and $w(O\phi) \geq \alpha$. In this project the idea was to formalize uncertainty about obligations. This can be understood as going from an uncertain perception of an obligation to a certain perception of an obligation. To express degrees of certainty and uncertainty we used $w()$ weight terms and to express obligations we used O the deontic modal operator. This means that for our purpose $w(O\phi)$ is the type of combination we wanted, and this is also exactly the way uncertainty/certainty about obligations was formalized. The idea to use τ came to mind when it was determined that a deontic model and world (D_s, w_s) were needed inside the probabilistic model to evaluate deontic formulas. By letting τ map pointed deontic models (D_s, w_s) to probabilistic states s it was possible to neatly fit everything needed for the logic in a tuple M. Furthermore the benefit of this method is that the deontic and probabilistic formulas constrain the logic exactly to those formulas of the type needed.

The formulas that are due to τ restricted are obligations about weight terms. They are prohibited due to the way τ functions but also and more fundamentally because the semantics is difficult to get right. The challenge with the semantics is to define it such that both $O(w(\theta) \geq \alpha)$ and $w(O\phi) \geq \alpha$ are interpretable in a meaningful way. Since the logic is meant to describe "uncertainty about obligations" we have to use that interpretation for O and $w()$. This means that the prohibited type of formula has to be read as an "obligation about uncertainty". To include such formulas in the logic there need to be proper "things" in the real world that can be described by it. An example could be to express that it is an obligation to be uncertain about another persons motives. This sentence is still rather difficult and leaves room for speculation of its viability. Furthermore the example does not fit well with the notion the logic is created for: uncertainty about obligations. Possible domains where rules occur involving uncertainty or probability are Chance Games and Quantum Mechanics. A possible interpretation of $O(w(\phi) \geq \alpha)$ could then be the theoretical probability that a process generates ie. what the probability of an event *should* be. With this I mean the probability to which the empirical probability of an event converges when obtaining ever more data of the event. In the case of throwing a dice the probabilities of each face converge to 1/6 when taking an infinite number of throws. This is what the probability of the event should be but not necessarily the probability that is measured after a finite number of dice throws.

When it is attempted to create a logic that does allow interaction between the deontic and probabilistic operator. The semantics of the operators should be the first class citizens, because it is difficult to find an interpretation that remains meaningful for both formulas. Even though constructing such a logic seems feasible (in [1] it is done with epistemic logic) from the perspective of models; thought it is difficult to get passed the semantic inflection that occurs when the operators change order. Also when considering the logic with regards to a domain it will be wise to first check whether the extra formula $O(w(\phi) \geq \alpha)$ is actually needed. This in regards to projection of this formula onto the now usable $w(O\phi) \geq \alpha$ type of formulas. In which case it is advisable to use a variant of $\mathfrak{PDL}$. Furthermore when giving semantics to O it should be considered what O really means is it an action or a rule that simply holds. Also it can be considered where the Deontic formulas actually come from i.e. does an agent create them or are they learned or uncovered.

Ultimately $\mathfrak{PDL}$can compare the weights given to rules. It does so by "importing" standard pointed deontic models –using a function called "τ"– into a probabilistic model and attributing probability to the formulas using a standard probability measure.

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